THE EQUALITY $I^2 = QI$ IN BUCHSBAUM RINGS

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Abstract. Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A$. Let Q be a parameter ideal in A. Let $I = Q : \mathfrak{m}$. The problem of when the equality $I^2 = QI$ holds true is explored. When A is a Cohen-Macaulay ring, this problem was completely solved by A. Corso, C. Huneke, C. Polini, and W. Vasconcelos [CHV, CP, CPV], while nothing is known when A is not a Cohen-Macaulay ring. The present purpose is to show that within a huge class of Buchsbaum local rings A the equality $I^2 = QI$ holds true for all parameter ideals Q. The result will supply [Y1, Y2] and [GN] with ample examples of ideals I, for which the Rees algebras $\mathrm{R}(I) = \bigoplus_{n \geq 0} I^n$, the associated graded rings $\mathrm{G}(I) = \mathrm{R}(I)/I\mathrm{R}(I)$, and the fiber cones $\mathrm{F}(I) = \mathrm{R}(I)/\mathfrak{m}\mathrm{R}(I)$ are all Buchsbaum rings with certain specific graded local cohomology modules. Two examples are explored. One is to show that $I^2 = QI$ may hold true for all parameter ideals Q in A, even though A is not a generalized Cohen-Macaulay ring, and the other one is to show that the equality $I^2 = QI$ may fail to hold for some parameter ideal Q in A, even though A is a Buchsbaum local ring with multiplicity at least three.

1. Introduction.

Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A$. Let Q be a parameter ideal in A and let $I = Q : \mathfrak{m}$. In this paper we will study the problem of when the equality $I^2 = QI$ holds true. K. Yamagishi [Y1, Y2] and the first author and K. Nishida [GN] have recently showed the Rees algebras $\mathrm{R}(I) = \bigoplus_{n \geq 0} I^n$, the associated graded rings $\mathrm{G}(I) = \mathrm{R}(I)/I\mathrm{R}(I)$, and the fiber cones $\mathrm{F}(I) = \mathrm{R}(I)/\mathfrak{m}\mathrm{R}(I)$ are all Buchsbaum rings with very specific graded local cohomology modules, if $I^2 = QI$ and the base rings A are Buchsbaum. Our results will supply [Y1, Y2] and [GN] with ample examples.

Our research dates back to the remarkable results of A. Corso, C. Huneke, C. Polini, and W. Vasconcelos [CHV, CP, CPV], who asserted that if A is a Cohen-Macaulay local ring, then the equality $I^2 = QI$ holds true for every parameter ideal Q in A, unless A is a regular local ring. Let $\mathfrak{a}^\sharp$ denote, for an ideal $\mathfrak{a}$ in A, the integral closure of $\mathfrak{a}$. Then

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their results are summarized into the following, in which the equivalence of assertions (2) and (3) are due to [G3, Theorem (3.1)]. The reader may consult [GH] for a simple proof of Theorem (1.1) with a slightly general form.

Theorem (1.1) ([CHV, CP, CPV]). *Let A be a Cohen-Macaulay ring with $\dim A = d$. Let Q be a parameter ideal in A and let $I = Q : \mathfrak{m}$. Then the following three conditions are equivalent to each other.*

(1) $I^2 \neq QI$.

(2) $Q = Q^\sharp$.

(3) *A is a regular local ring which contains a regular system $a_1, a_2, \cdots, a_d$ of parameters such that $Q = (a_1, \cdots, a_{d-1}, a_d^q)$ for some $1 \leq q \in \mathbb{Z}$.*

Hence $I^2 = QI$ for every parameter ideal Q in A, unless A is a regular local ring.

Our purpose is to generalize Theorem (1.1) to local rings A which are not necessarily Cohen-Macaulay. Since the notion of Buchsbaum ring is a straightforward generalization of that of Cohen-Macaulay ring, it seems quite natural to expect that the equality $I^2 = QI$ still holds true also in Buchsbaum rings. This is, nevertheless, in general not true and a counterexample is already explored by [CP]. Let $A = k[[X, Y]]/(X^2, XY)$ where $k[[X, Y]]$ denotes the formal power series ring in two variables over a field k and let x, y be the reduction of X, Y mod the ideal (X^2, XY). Let $Q = (y^3)$ and put $I = Q : \mathfrak{m}$. Then $I = (x, y^2)$ and $I^2 \neq QI$ ([CP, p. 231]). However, the ideal Q is actually *not* the reduction of I and the multiplicity $\mathrm{e}(A)$ of A is 1. The Buchsbaum local ring A is *almost* a DVR in the sense that $A/(x)$ is a DVR and $\mathfrak{m}{\cdot}x = (0)$. Added to it, with no difficulty one is able to check that for a given parameter ideal Q in A, the equality $I^2 = QI$ holds true if and only if $Q \not\subseteq \mathfrak{m}^2$. For these reasons this example looks rather dissatisfactory, and we shall provide in this paper more drastic counterexamples. Nonetheless, the example [CP, p. 231] was invaluable for the authors to settle their starting point towards the present research. For instance, it strongly suggests that for the study of the equality $I^2 = QI$ we first of all have to find the conditions under which Q is a reduction of I, and the condition $\mathrm{e}(A) \neq 1$ might play a certain role in it. Any DVR contains no parameter ideals Q for which the equality $I^2 = QI$ holds true, while as the example shows, non-Cohen-Macaulay Buchsbaum local rings with $\mathrm{e}(A) = 1$ could contain somewhat ampler parameter ideals Q for which the equality $I^2 = QI$ holds true.

Our problem is, therefore, divided into two parts. One is to clarify the condition under which Q is a reduction of I and the other one is to evaluate, when $I \subseteq Q^\sharp$, the reduction number

$$\mathrm{r}_Q(I) = \min\{0 \leq n \in \mathbb{Z} \mid I^{n+1} = QI^n\}$$

of I with respect to Q. As we shall quickly show in this paper, one always has that

$I \subseteq Q^\sharp$, unless $\mathrm{e}(A) = 1$. In contrast, the second part of our problem is in general quite subtle and unclear, as we will eventually show in this paper. We shall, however, show that within a huge class of Buchsbaum local rings A, the equality $I^2 = QI$ holds true for every parameter ideal Q in A.

Let us now state more precisely our main results, explaining how this paper is organized. In Section 2 we will prove that if $\mathrm{e}(A) > 1$, then $I = Q : \mathfrak{m} \subseteq Q^\sharp$ for every parameter ideal Q in A. Hence Q is a *minimal* reduction of I, satisfying the equality $\mathfrak{m}I^n = \mathfrak{m}Q^n$ for all $\in \mathbb{Z}$ (Proposition (2.3)). Our proof is based on the induction on $d = \dim A$, and the difficulty that we meet whenever we will check whether $I^2 = QI$ is caused by the wild behavior of the socle $(0) : \mathfrak{m}$ in A. So, in Section 2, we shall closely explain the method how to control the socle $(0) : \mathfrak{m}$ in our context (Lemma (2.4)). The main results of the section are Theorem (2.1) and Corollary (2.13), which assert that every unmixed local ring A with $\dim A \geq 2$ contains infinitely many parameter ideals Q, for which the equality $I^2 = QI$ holds true.

In Section 3 we are concentrated to the case where A is a Buchsbaum local ring. Let A be a Buchsbaum local ring with $d = \dim A \geq 1$ and let $x_1, x_2, \cdots, x_d$ be a system of parameters in A. Let $n_i \geq 1$ $(1 \leq i \leq d)$ be integers and put $Q = (x_1^{n_1}, x_2^{n_2}, \cdots, x_d^{n_d})$. We will then show that $I^2 = QI$ if $\mathrm{e}(A) > 1$ and if $n_i \geq 2$ for some $1 \leq i \leq d$ (Theorem (3.3)). Consequently, in a Buchsbaum local ring A of the form $A = B/(f^n)$ where $n \geq 2$ and f is a parameter in a Buchsbaum local ring B, the equality $I^2 = QI$ holds true for every parameter ideal Q (Corollary (3.7)).

Let $\mathrm{r}(A) = \sup_Q \ \ell_A((Q : \mathfrak{m})/Q)$ where Q runs over parameter ideals in A, which we call the Cohen-Macaulay type of A. Then, thanks to Theorem (2.5) of [GSu], one has the equality

$$\mathrm{r}(A) = \sum_{i=0}^{d-1} \binom{d}{i} h^i(A) + \mu_{\hat{A}}(\mathrm{K}_{\hat{A}})$$

for every Buchsbaum local ring A with $d = \dim A \geq 1$, where $h^i(A) = \ell_A(\mathrm{H}^i_\mathfrak{m}(A))$ denotes the length of the i $\underline{th}$ local cohomology module of A with respect to $\mathfrak{m}$ and $\mu_{\hat{A}}(\mathrm{K}_{\hat{A}})$ denotes the number of generators for the canonical module $\mathrm{K}_{\hat{A}}$ of the $\mathfrak{m}$-adic completion $\widehat{A}$ of A. Accordingly, one has $\ell_A((Q : \mathfrak{m})/Q) \leq \mathrm{r}(A)$ in general, and if furthermore $\ell_A((Q : \mathfrak{m})/Q) = \mathrm{r}(A)$, then the equality $I^2 = QI$ holds true for the ideal $I = Q : \mathfrak{m}$, provided A is a Buchsbaum local ring with $\mathrm{e}(A) > 1$ (Theorem (3.9)). Consequently, if A is a Buchsbaum local ring with $\mathrm{e}(A) > 1$ and the index $\ell_A((Q : \mathfrak{m})/Q)$ of reducibility of Q is independent of the choice of a parameter ideal Q in A, the equality $I^2 = QI$ then holds true for all parameter ideals Q in A. This result seems to account well for the reason why Theorem (1.1) holds true for Cohen-Macaulay rings A. In Section 3 we shall also show that for a Buchsbaum local ring A, there exists an integer $\ell \gg 0$ such that

the equality $\mathrm{r}(A) = \ell_A((Q : \mathfrak{m})/Q)$ holds true for all parameter ideals $Q \subseteq \mathfrak{m}^\ell$ (Theorem (3.11)). Thus, inside Buchsbaum local rings A with $d = \dim A \geq 2$, the parameter ideals Q satisfying the equality $I^2 = QI$ are in the majority. In the forthcoming paper [GSa] we will also prove that the equality $I^2 = QI$ holds true for all parameter ideals Q in a Buchsbaum local ring A with $\mathrm{e}(A) = 2$ and $\operatorname{depth} A > 0$.

In Section 4 we will give an effective evaluation of the reduction numbers $\mathrm{r}_Q(I)$ in the case where A is a Buchsbaum local ring with $\dim A = 1$ and $\mathrm{e}(A) > 1$ (Theorem (4.1)). The evaluation is sharp, as we will show with an example. The authors do not know whether there exist some uniform bounds of $\mathrm{r}_Q(I)$ also in higher dimensional cases.

It is somewhat surprising to see that the equality $I^2 = QI$ may hold true for *all* parameter ideals Q in A, even though A is not a generalized Cohen-Macaulay ring. In Section 5 we will explore one example satisfying this property (Theorem (5.3)). In contrast, the equality $I^2 = QI$ does in general not hold true, even though A is a Buchsbaum local ring with $\mathrm{e}(A) > 1$. In Section 5 we shall also explore one more example of dimension 1 (Theorem (5.17)), giving complete criteria of the equality $I^2 = QI$ for parameter ideals Q in the example.

We are now entering the very details. Before that, let us fix again our standard notation. Throughout, let $(A, \mathfrak{m})$ be a Noetherian local ring with $d = \dim A$. We denote by $\mathrm{e}(A) = \mathrm{e}^0_\mathfrak{m}(A)$ the multiplicity of A with respect to the maximal ideal $\mathfrak{m}$. Let $\mathrm{H}^i_\mathfrak{m}(*)$ denote the local cohomology functor with respect to $\mathfrak{m}$. We denote by $\ell_A(*)$ and $\mu_A(*)$ the length and the number of generators, respectively. Let $\mathfrak{a}^\sharp$ denote for an ideal $\mathfrak{a}$ in A the integral closure of $\mathfrak{a}$. Let $Q = (x_1, x_2, \cdots, x_d)$ be a parameter ideal in A and, otherwise specified, we denote by I the ideal $Q : \mathfrak{m}$. Let $\operatorname{Min} A$ be the set of minimal prime ideals in A. Let $\widehat{A}$ denote the $\mathfrak{m}$-adic completion of A.

2. A theorem for general local rings.

The goal of this section is the following.

Theorem (2.1). *Let A be a Noetherian local ring with $d = \dim A \geq 2$. Assume that A is a homomorphic image of a Gorenstein local ring and $\dim A/\mathfrak{p} = d$ for all $\mathfrak{p} \in \operatorname{Ass} A$. Then A contains a system $a_1, a_2, \cdots, a_d$ of parameters such that for all integers $n_i \geq 1$ $(1 \leq i \leq d)$ the equality $I^2 = QI$ holds true, where*

$$Q = (a_1^{n_1}, a_2^{n_2}, \cdots, a_d^{n_d}) \quad \textit{and} \quad I = Q : \mathfrak{m}.$$

To prove Theorem (2.1) we need some preliminary steps. Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A \geq 0$. Let Q be a parameter ideal in A. We put $I = Q : \mathfrak{m}$. We begin with the following.

Lemma (2.2). *Suppose that $d \geq 1$. Then* $\mathrm{e}(A) = 1$ *if* $\mathfrak{m}I \not\subseteq \mathfrak{m}Q$.

Proof. We may assume $I \neq A$. Let $W = \mathrm{H}^0_{\mathfrak{m}}(A)$ and $B = A/W$. If $d = 1$, then $Q = \mathfrak{m}I$, since Q is a principal ideal. Let $Q = (a)$, $\overline{\mathfrak{m}} = \mathfrak{m}B$, and $\overline{I} = IB$. Let $\overline{a} = a \bmod W$. Then, since $(\overline{a}) = \overline{\mathfrak{m}}{\cdot}\overline{I}$ and $\overline{a}$ is a non-zerodivisor in the Cohen-Macaulay local ring B, the maximal ideal $\overline{\mathfrak{m}}$ is invertible, so that B is a DVR; hence $\mathrm{e}(B) = \mathrm{e}(A) = 1$. Suppose that $d \geq 2$ and that our assertion holds true for $d-1$. We choose $a_d \in \mathfrak{m}I$ so that $a_d \notin \mathfrak{m}Q$, and then write $Q = (a_1, \cdots, a_{d-1}, a_d)$. Let $\overline{A} = A/(a_1)$, $\overline{\mathfrak{m}} = \mathfrak{m}/(a_1)$, $\overline{Q} = Q/(a_1)$, and $\overline{I} = I/(a_1)$. Let $\overline{a_i} = a_i \bmod (a_1)$ $(2 \leq i \leq d)$. Then $\overline{Q} = (\overline{a_2}, \cdots, \overline{a_d})$ is a parameter ideal in $\overline{A}$ and $\overline{I} = \overline{Q} : \overline{\mathfrak{m}}$. We have $\overline{\mathfrak{m}}\overline{I} \not\subseteq \overline{\mathfrak{m}}\overline{Q}$, since $\overline{a_d} \notin \overline{\mathfrak{m}}\overline{Q}$. Hence $\mathrm{e}(\overline{A}) = 1$ by the hypothesis on d, so that $\mathrm{e}(A) = 1$ as well. $\square$

Proposition (2.3). *Suppose that* $\mathrm{e}(A) > 1$. *Then* $I \subseteq Q^\sharp$ *and* $\mathfrak{m}I^n = \mathfrak{m}Q^n$ *for all* $n \in \mathbb{Z}$.

Proof. We may assume that $d \geq 1$. Let $W = \mathrm{H}^0_{\mathfrak{m}}(A)$ and put $B = A/W$. Then $\mathfrak{m}B{\cdot}IB \subseteq \mathfrak{m}B{\cdot}QB$, since $\mathfrak{m}I \subseteq \mathfrak{m}Q$ by Lemma (2.2). Thus IB is integral over QB, because the ideal $\mathfrak{m}B$ contains a non-zerodivisor of B (recall that $\operatorname{depth} B \geq 1$). Consequently, since $W \subseteq \sqrt{(0)}$, I is integral over Q, so that Q is a minimal reduction of I. Since $\mathfrak{m}I \cap Q = \mathfrak{m}Q$, we have that $\mathfrak{m}I = \mathfrak{m}Q$, and hence $\mathfrak{m}I^n = \mathfrak{m}Q^n$ for all $n \in \mathbb{Z}$. $\square$

The assertion that $I \subseteq Q^\sharp$ is in general no longer true, unless $\mathrm{e}(A) > 1$ (see Theorem (1.1)). When A is not a Cohen-Macaulay ring, the result is more complicated, as we shall explore in Section 5.

The following result plays a key role throughout this paper as well as in the proof of Theorem (2.1).

Lemma (2.4). *Let R be any commutative ring. Let M, L, and W be ideals in R and let $x \in M$. Assume that $L : x^2 = L : x$ and $xW = (0)$. Then*

$$(L + (x^n) + W) : M = [(L + W) : M] + [(L + (x^n)) : M]$$

for all $n \geq 2$. If $L : x = L : M$, we furthermore have that

$$(L + (x^n) + W) : M = (L + (x^n)) : M$$

for all $n \geq 2$.

Proof. We have $L : x^\ell = L : x$ and $[L + (x^\ell)] \cap [L : (x^\ell)] = L$ for all $\ell \geq 1$, since $L : x^2 = L : x$. Let $\varphi \in (L + (x^n) + W) : M$ and write $x\varphi = \ell + x^n y + w$, where $\ell \in L$, $y \in R$, and $w \in W$. Let $z = \varphi - x^{n-1}y$. Then since $x^2\varphi = x\ell + x^{n+1}y$, we have

$$z = \varphi - x^{n-1}y \in L : x^2 = L : x. \tag{2.5}$$

Let $\alpha \in M$ and write $\alpha\varphi = \ell_1 + x^n y_1 + w_1$ with $\ell_1 \in L$, $y_1 \in R$, and $w_1 \in W$. Then because

$$\alpha\varphi = \ell_1 + x^n y_1 + w_1 = \alpha z + x^{n-1}(\alpha y)$$

we get $\alpha z - w_1 \in [L + (x^{n-1})] \cap [L : x] \subseteq L$ (recall that $w_1 \in W \subseteq L : x$), whence

$$z \in (L + W) : M \subseteq (L + (x^n) + W) : M$$

so that we also have $x^{n-1}y = \varphi - z \in (L + (x^n) + W) : M$. Let $\alpha \in M$ and write $x^{n-1}(\alpha y) = \ell_2 + x^n y_2 + w_2$ with $\ell_2 \in L$, $y_2 \in R$, and $w_2 \in W$. Then $x^n(\alpha y) = x\ell_2 + x^{n+1}y_2$ and $\alpha y - xy_2 \in L : x^n = L : x$. Hence $y \in ((L : x) + (x)) : M$, so that $x^{n-1}y \in (L + (x^n)) : M$ since $n \geq 2$. Thus

$$\varphi = z + x^{n-1}y \in [(L + W) : M] + [(L + (x^n)) : M].$$

If $L : x = L : M$ in addition, we get $z \in L : M$ by (2.5), whence

$$\varphi = z + x^{n-1}y \in [L : M] + [(L + (x^n)) : M] = (L + (x^n)) : M$$

as is claimed. $\square$

Let R be a commutative ring and $x_1, x_2, \cdots, x_s \in R$ ($s \geq 1$). Then $x_1, x_2, \cdots, x_s$ is called a d-sequence in R, if

$$(x_1, \cdots, x_{i-1}) : x_i = (x_1, \cdots, x_{i-1}) : x_i x_j$$

whenever $1 \leq i \leq j \leq s$. We say that $x_1, x_2, \cdots, x_s$ forms a strong d-sequence in R, if $x_1^{n_1}, x_2^{n_2}, \cdots, x_s^{n_s}$ is a d-sequence in R for all integers $n_i \geq 1$ ($1 \leq i \leq s$). See [H] for basic but deep results on d-sequences, which we shall freely use in this paper. For example, if $x_1, x_2, \cdots, x_s$ is a d-sequence in R, then

$$(2.6) \qquad \begin{aligned} (x_1, \cdots, x_{i-1}) : x_i^2 &= (x_1, \cdots, x_{i-1}) : x_i \\ &= (x_1, \cdots, x_{i-1}) : (x_1, x_2, \cdots, x_s) \end{aligned}$$

for all $1 \leq i \leq s$. Also one has the equality

$$(2.7) \qquad ((x_1, \cdots, x_{i-1}) : x_i) \cap (x_1, x_2, \cdots, x_s)^n = (x_1, \cdots, x_{i-1}) \cdot (x_1, x_2, \cdots, x_s)^{n-1}$$

for all integers $1 \leq i \leq s$ and $1 \leq n \in \mathbb{Z}$.

The following result is due to N. T. Cuong.

Proposition (2.8) ([C, Theorem 2.6]). *Let A be a Noetherian local ring with $d = \dim A \geq 1$. Assume that A is a homomorphic image of a Gorenstein local ring and that $\dim A/\mathfrak{p} = d$ for all $\mathfrak{p} \in \mathrm{Ass}\ A$. Then A contains a system $x_1, x_2, \cdots, x_d$ of parameters which forms a strong d-sequence.*

We will apply the following result to strong d-sequences of Cuong.

Proposition (2.9). *Let R be a commutative ring and let $x_1, x_2, \cdots, x_s \in R$ $(s \geq 1)$. Let $Q = (x_1, x_2, \cdots, x_s)$ and $W = (0) : Q$. Let M be an ideal in R such that $Q \subseteq M$. Assume that $x_1, x_2, \cdots, x_s$ is a strong d-sequence in R. Then*

$$[(x_1^{n_1}, x_2^{n_2}, \cdots, x_s^{n_s}) + W] : M = W + [(x_1^{n_1}, x_2^{n_2}, \cdots, x_s^{n_s}) : M]$$

for all integers $n_i \geq 2$ $(1 \leq i \leq s)$.

Proof. We put $L = (x_1^{n_1}, \cdots, x_{s-1}^{n_{s-1}})$, $x = x_s$, and $n = n_s$. Then $L : x^2 = L : x$, $x \in M$, and $xW = (0)$. Hence by Lemma (2.4) we get

$$[L + (x^n) + W] : M = [(L + W) : M] + [(L + (x^n)) : M]. \tag{2.10}$$

Notice that $W : M = W$. (For, if $\varphi \in W : M$, then $x_1\varphi \in W$ so that $x_1^2\varphi = 0$, whence $\varphi \in (0) : x_1^2 = (0) : x_1 = W$; cf (2.6).) Our assertion is obviously true when $s = 1$. Suppose that $s \geq 2$ and that our assertion holds true for $s - 1$. Then, since $x_1, x_2, \cdots, x_{s-1}$ is a strong d-sequence in R and $W = (0) : x_1 = (0) : (x_1, \cdots, x_{s-1})$ by (2.6), by the hypothesis on s we readily get that

$$(L + W) : M = W + (L : M)$$

whence by (2.10)

$$\begin{aligned}
[(x_1^{n_1}, x_2^{n_2}, \cdots, x_s^{n_s}) + W] : M &= [(L + (x^n) + W)] : M \\
&= [W + (L : M)] + [(L + (x^n)) : M] \\
&= W + [(L + (x^n)) : M] \\
&= W + [(x_1^{n_1}, x_2^{n_2}, \cdots, x_s^{n_s}) : M]
\end{aligned}$$

as is claimed. □

We are now back to local rings.

Corollary (2.11). *Let $x_1, x_2, \cdots, x_d$ be a system of parameters in a Noetherian local ring A with $d = \dim A \geq 1$ and assume that $x_1, x_2, \cdots, x_d$ forms a strong d-sequence.*

Let $n_i \geq 2$ $(1 \leq i \leq d)$ be integers and put $Q = (x_1^{n_1}, x_2^{n_2}, \cdots, x_d^{n_d})$. Then $I^2 = QI$ if $\mathrm{e}(A) > 1$, *where* $I = Q : \mathfrak{m}$.

Proof. Let $W = \mathrm{H}^0_{\mathfrak{m}}(A)$. Then $W = (0) : x_1 = (0) : (x_1, x_2, \cdots, x_d)$. (For, if $\varphi \in W$, then $x_1^n \varphi = 0$ for some $n \gg 0$, whence $\varphi \in (0) : x_1^n = (0) : x_1 = (0) : (x_1, x_2, \cdots, x_d)$; cf (2.6).) Let $B = A/W$. Then since

$$(Q + W) : \mathfrak{m} = W + (Q : \mathfrak{m}) = W + I$$

by Proposition (2.9), we get $IB = QB : \mathfrak{m}B$. If $d = 1$, then $(IB)^2 = QB{\cdot}IB$ by Theorem (1.1), because B is a Cohen-Macaulay ring with $\mathrm{e}(B) = \mathrm{e}(A) > 1$. Hence $I^2 \subseteq QI + W$, so that we have $I^2 = QI$, because

$$W \cap Q \subseteq [(0) : (x_1)] \cap (x_1, x_2, \cdots, x_d) = (0)$$

(cf. (2.7)). Suppose that $d \geq 2$ and that our assertion holds true for $d - 1$. Let $a_i = x_i^{n_i}$ $(1 \leq i \leq d)$ and put $\overline{A} = A/(a_1)$ and $\overline{I} = I/(a_1)$. For each $c \in A$ let $\overline{c}$ denote the reduction of c mod (a_1). Then, since $\mathrm{e}(\overline{A}) > 1$ and the system $\overline{x_2}, \cdots, \overline{x_d}$ of parameters for $\overline{A}$ forms by definition a strong d-sequence in $\overline{A}$, thanks to the hypothesis on d, we get $\overline{I}^2 = (\overline{a_2}, \cdots, \overline{a_d})\overline{I}$. Hence $I^2 \subseteq (a_2, \cdots, a_d)I + (a_1)$ and so $I^2 = (a_2, \cdots, a_d)I + [(a_1) \cap I^2]$.

We then need the following.

Claim (2.12). $(a_1) \cap I^2 = a_1 I$.

Proof of Claim (2.12). Let $\varphi \in (a_1) \cap I^2$ and write $\varphi = a_1 y$ with $y \in A$. Let $\alpha \in \mathfrak{m}$. Then $\alpha\varphi = a_1(\alpha y) \in Q^2$ since $\mathfrak{m}I^2 \subseteq Q^2$ (cf. (2.3)). Consequently $a_1(\alpha y) \in (a_1) \cap Q^2 = a_1 Q$ (cf. (2.7)). Hence $\alpha y - q \in (0) : a_1 = (0) : x_1 = W$ for some $q \in Q$. Thus

$$y \in (Q + W) : \mathfrak{m} = W + I$$

so that $\varphi = a_1 y \in a_1 I$. Thus $(a_1) \cap I^2 = a_1 I$, which completes the proof of Corollary (2.11) and Claim (2.12) as well. $\square$

We are now ready to prove Theorem (2.1).

Proof of Theorem (2.1). Choose a system $y_1, y_2, \cdots, y_d$ of parameters for A that forms a strong d-sequence in A (this choice is possible; cf. Proposition (2.8)). Let $x_i = y_i^2$ $(1 \leq i \leq d)$. Then the sequence $x_1, x_2, \cdots, x_d$ is still a strong d-sequence in A. If $\mathrm{e}(A) > 1$, then by Corollary (2.11) $I^2 = QI$ for the parameter ideals $Q = (x_1^{n_1}, x_2^{n_2}, \cdots, x_d^{n_d})$ with $n_i \geq 1$. Suppose that $\mathrm{e}(A) = 1$. Then A is a regular local ring, since A is unmixed, i.e., $\dim \widehat{A}/\mathfrak{p} = d$ for all $\mathfrak{p} \in \mathrm{Ass}\,\widehat{A}$. Hence $I^2 = QI$ by Theorem (1.1) since $Q \subseteq \mathfrak{m}^2$, which completes the proof of Theorem (2.1). $\square$

Since every parameter ideal $\widehat{Q}$ in $\widehat{A}$ has the form $\widehat{Q} = Q\widehat{A}$ with Q a parameter ideal in A, from Theorem (2.1) we readily get the following.

Corollary (2.13). *Let A be a Noetherian local ring with $d = \dim A \geq 2$. Assume that A is unmixed, that is* $\dim \widehat{A}/\mathfrak{p} = d$ *for all* $\mathfrak{p} \in \operatorname{Ass} \widehat{A}$. *Then A contains infinitely many parameter ideals Q, for which the equality $I^2 = QI$ holds true, where $I = Q : \mathfrak{m}$.*

Let A be a Noetherian local ring with $d = \dim A \geq 1$. Then we say that A is a generalized Cohen-Macaulay ring (or simply, A has FLC), if all the local cohomology modules $\mathrm{H}^i_{\mathfrak{m}}(A)$ $(i \neq d)$ are finitely generated A-modules. This condition is equivalent to saying that there exists an integer $\ell \gg 0$ such that every system of parameters contained in $\mathfrak{m}^\ell$ forms a d-sequence ([CST]). Consequently, when A is a generalized Cohen-Macaulay ring, every system of parameters contained in $\mathfrak{m}^\ell$ forms a strong d-sequence in any order, so that by Corollary (2.11) our local ring A contains numerous parameter ideals Q for which the equality $I^2 = QI$ holds true, unless $\mathrm{e}(A) = 1$. Nevertheless, even though A is a generalized Cohen-Macaulay ring with $\mathrm{e}(A) > 1$, it remains subtle whether $I^2 = QI$ for every parameter ideal Q contained in $\mathfrak{m}^\ell$ $(\ell \gg 0)$. In the next section we shall study this problem in the case where A is a Buchsbaum ring.

3. Buchsbaum local rings.

Let A be a Noetherian local ring and $d = \dim A \geq 1$. Then A is said to be a Buchsbaum ring, if the difference

$$\mathrm{I}(A) = \ell_A(A/Q) - \mathrm{e}^0_Q(A)$$

is independent of the particular choice of a parameter ideal Q in A and is an invariant of A, where $\mathrm{e}^0_Q(A)$ denotes the multiplicity of A with respect to Q. The condition is equivalent to saying that every system $x_1, x_2, \cdots, x_d$ of parameters for A forms a weak A-sequence, that is the equality

$$(x_1, \cdots, x_{i-1}) : x_i = (x_1, \cdots, x_{i-1}) : \mathfrak{m}$$

holds true for all $1 \leq i \leq d$ (cf. [SV1]). Hence every system of parameters for a Buchsbaum local ring forms a strong d-sequence in any order. Cohen-Macaulay local rings A are Buchsbaum rings with $\mathrm{I}(A) = 0$, and vice versa. In this sense the notion of Buchsbaum ring is a natural generalization of that of Cohen-Macaulay ring.

If A is a Buchsbaum ring, then all the local cohomology modules $\mathrm{H}^i_{\mathfrak{m}}(A)$ $(i \neq d)$ are killed by the maximal ideal $\mathfrak{m}$, and one has the equality

$$\mathrm{I}(A) = \sum_{i=0}^{d-1} \binom{d-1}{i} h^i(A)$$

where $h^i(A) = \ell_A(\mathrm{H}^i_{\mathfrak{m}}(A))$ for $0 \leq i \leq d-1$ (cf. [SV2, Chap. I, (2.6)]). It was proven by [G1, Theorem (1.1)] that for given integers d and $h_i \geq 0$ $(0 \leq i \leq d-1)$ there exists a

Buchsbaum local ring $(A, \mathfrak{m})$ such that $\dim A = d$ and $h^i(A) = h_i$ for all $0 \le i \le d-1$. One may also choose the Buchsbaum ring A so that A is an integral domain (resp. a normal domain), if $h_0 = 0$ (resp. $d \ge 2$ and $h_0 = h_1 = 0$). See the book [SV2] for the basic results on Buchsbaum rings and modules.

Let A be a Buchsbaum local ring with $d = \dim A \ge 1$ and let

$$\mathrm{r}(A) = \sup_Q \ \ell_A((Q : \mathfrak{m})/Q)$$

where Q runs over parameter ideals in A. Then one has the equality

$$\mathrm{r}(A) = \sum_{i=0}^{d-1} \binom{d}{i} h^i(A) + \mu_{\hat{A}}(\mathrm{K}_{\hat{A}})$$

where $\mathrm{K}_{\hat{A}}$ denotes the canonical module of $\widehat{A}$ (cf. [GSu, Theorem (2.5)]). In particular $\mathrm{r}(A) < \infty$.

We need the following, which is implicitly known by [GSu]. We note a sketch of proof for the sake of completeness.

Proposition (3.1). *Let A be a Buchsbaum local ring with $d = \dim A \ge 2$. Then one has the inequality* $\mathrm{r}(A/(a)) \le \mathrm{r}(A)$ *for every $a \in \mathfrak{m}$ such that* $\dim A/(a) = d-1$.

Proof. Let $B = A/(a)$. Then since $\mathfrak{m}{\cdot}[(0) : a] = (0)$, from the exact sequence

$$0 \to (0) : a \to A \xrightarrow{a} A \to B \to 0$$

we get a long exact sequence

$$\begin{aligned} 0 \to (0) : a \to \mathrm{H}^0_{\mathfrak{m}}(A) &\xrightarrow{a} \mathrm{H}^0_{\mathfrak{m}}(A) \to \mathrm{H}^0_{\mathfrak{m}}(B) \\ \to \mathrm{H}^1_{\mathfrak{m}}(A) &\xrightarrow{a} \mathrm{H}^1_{\mathfrak{m}}(A) \to \mathrm{H}^1_{\mathfrak{m}}(B) \\ \cdots & \\ \to \mathrm{H}^i_{\mathfrak{m}}(A) &\xrightarrow{a} \mathrm{H}^i_{\mathfrak{m}}(A) \to \mathrm{H}^i_{\mathfrak{m}}(B) \\ \cdots & \\ \to \mathrm{H}^d_{\mathfrak{m}}(A) &\xrightarrow{a} \mathrm{H}^d_{\mathfrak{m}}(A) \to \mathrm{H}^d_{\mathfrak{m}}(B) \to \cdots \end{aligned}$$

of local cohomology modules, which splits into the following short exact sequences

$$(3.2) \qquad 0 \to \mathrm{H}^i_{\mathfrak{m}}(A) \to \mathrm{H}^i_{\mathfrak{m}}(B) \to \mathrm{H}^{i+1}_{\mathfrak{m}}(A) \to 0 \quad (0 \le i \le d-2) \quad \text{and}$$

$$(3.3) \qquad 0 \to \mathrm{H}^{d-1}_{\mathfrak{m}}(A) \to \mathrm{H}^{d-1}_{\mathfrak{m}}(B) \to [(0) :_{\mathrm{H}^d_{\mathfrak{m}}(A)} a] \to 0,$$

because $a{\cdot}\mathrm{H}^i_{\mathfrak{m}}(A) = (0)$ for all $i \neq d$. Hence $h^i(B) = h^i(A) + h^{i+1}(A)$ $(0 \leq i \leq d-2)$ by (3.2). Apply the functor $\mathrm{Hom}_A(A/\mathfrak{m}, *)$ to sequence (3.3) and we have the exact sequence

$$0 \to \mathrm{H}^{d-1}_{\mathfrak{m}}(A) \to [(0) :_{\mathrm{H}^{d-1}_{\mathfrak{m}}(B)} \mathfrak{m}] \to [(0) :_{\mathrm{H}^{d}_{\mathfrak{m}}(A)} \mathfrak{m}]. \tag{3.4}$$

Hence

$$\begin{aligned}
\mathrm{r}(B) &= \sum_{i=0}^{d-2} \binom{d-1}{i} h^i(B) + \mu_{\hat{B}}(\mathrm{K}_{\hat{B}}) \\
&= \sum_{i=0}^{d-2} \binom{d-1}{i} \{h^i(A) + h^{i+1}(A)\} + \mu_{\hat{B}}(\mathrm{K}_{\hat{B}}) \\
&= \left\{\sum_{i=0}^{d-1} \binom{d}{i} h^i(A) - h^{d-1}(A)\right\} + \mu_{\hat{B}}(\mathrm{K}_{\hat{B}}) \\
&\leq \left\{\sum_{i=0}^{d-1} \binom{d}{i} h^i(A) - h^{d-1}(A)\right\} + \left\{h^{d-1}(A) + \mu_{\hat{A}}(\mathrm{K}_{\hat{A}})\right\} \quad (\text{by } \ (3.4)) \\
&= \sum_{i=0}^{d-1} \binom{d}{i} h^i(A) + \mu_{\hat{A}}(\mathrm{K}_{\hat{A}}) \\
&= \mathrm{r}(A)
\end{aligned}$$

as is claimed. $\square$

For the rest of this section, otherwise specified, let A be a Buchsbaum local ring and $d = \dim A \geq 1$. Let $W = \mathrm{H}^0_{\mathfrak{m}}(A)$ $(= (0) : \mathfrak{m})$.

To begin with we note the following.

Lemma (3.5). *Let $x_1, x_2, \cdots, x_d$ be a system of parameters for A. Let $n_i \geq 1$ be integers and put $Q = (x_1^{n_1}, x_2^{n_2}, \cdots, x_d^{n_d})$. Then $(Q + W) : \mathfrak{m} = Q : \mathfrak{m}$ if $n_i \geq 2$ for some $1 \leq i \leq d$.*

Proof. We may assume $n_d \geq 2$. Let $L = (x_1^{n_1}, \cdots, x_{d-1}^{n_{d-1}})$ and $x = x_d$. Then $L : x^2 = L : x = L : \mathfrak{m}$ and $xW = (0)$, since A is a Buchsbaum ring. Hence $(Q+W) : \mathfrak{m} = Q : \mathfrak{m}$ by Lemma (2.4), because $W = (0) : \mathfrak{m} \subseteq Q : \mathfrak{m}$. $\square$

Theorem (3.6). *Let $x_1, x_2, \cdots, x_d$ be a system of parameters for A and put $Q = (x_1^{n_1}, x_2^{n_2}, \cdots, x_d^{n_d})$ with $n_i \geq 1$ $(1 \leq i \leq d)$. Let $I = Q : \mathfrak{m}$. Then $I^2 = QI$ if $\mathrm{e}(A) > 1$ and $n_i \geq 2$ for some $1 \leq i \leq d$.*

Proof. Let $n_d \geq 2$. By Corollary (2.11) we may assume that $d \geq 2$ and that our assertion holds true for $d-1$. Let $a_i = x_i^{n_i}$ $(1 \leq i \leq d)$ and put $\overline{A} = A/(a_1)$. Then $x_2, \cdots, x_d$ forms

a system of parameters in the Buchsbaum local ring $\overline{A}$. Because $\mathrm{e}(\overline{A}) > 1$ and $n_d \geq 2$, by the hypothesis on d we get that $\overline{I}^2 = (\overline{a_2}, \cdots, \overline{a_d})\overline{I}$ in $\overline{A}$, where $\overline{a_i}$ denotes the reduction of a_i mod (a_1) and $\overline{I} = I/(a_1)$. Hence $I^2 \subseteq (a_2, \cdots, a_d)I + (a_1)$. Since $(Q + W) : \mathfrak{m} = I$ by Lemma (3.5), similarly as in the proof of Claim (2.12) we get $(a_1) \cap I^2 = a_1 I$, whence $I^2 = QI$ as is claimed. $\square$

In Corollary (2.11) one needs the assumption that $n_i \geq 2$ for *all* $1 \leq i \leq d$. In contrast, if A is a Buchsbaum local ring, that is the case of Theorem (3.6), this assumption is weakened so that $n_i \geq 2$ for *some* $1 \leq i \leq d$. Unfortunately the assumption in Theorem (3.6) is in general not superfluous, as we will show in Sections 4 and 5.

The following is an immediate consequence of Theorem (3.6).

Corollary (3.7). *Let $(R, \mathfrak{n})$ be a Buchsbaum local ring with $\dim R \geq 2$ and $\mathrm{e}(R) > 1$. Choose $f \in \mathfrak{n}$ so that $\dim R/(f) = \dim R - 1$ and put $A = R/(f^n)$ with $n \geq 2$. Then the equality $I^2 = QI$ holds true for every parameter ideal Q in A, where $I = Q : \mathfrak{m}$.*

Let us note one more consequence.

Corollary (3.8). *Let $x_1, x_2, \cdots, x_d$ be a system of parameters in a Buchsbaum local ring A with $d = \dim A \geq 2$ and let $Q = (x_1^{n_1}, x_2^{n_2}, \cdots, x_d^{n_d})$ with $n_i \geq 1$ $(1 \leq i \leq d)$. Then $I^2 = QI$ if $n_i, n_j \geq 2$ for some $1 \leq i, j \leq d$ with $i \neq j$.*

Proof. Thanks to Theorem (3.6) we may assume that $\mathrm{e}(A) = 1$. Let $B = A/W$. Then B is a regular local ring with $\dim B = d \geq 2$, because $\mathrm{e}(B) = 1$ and B is unmixed (cf. [CST]). We have $\ell_B((QB + \mathfrak{m}^2 B)/\mathfrak{m}^2 B) \leq d - 2$, since $x_i^{n_i}, x_j^{n_j} \in \mathfrak{m}^2$. Therefore $(IB)^2 = (QB){\cdot}(IB)$ by Theorem (1.1), because $IB = QB : \mathfrak{m}B$ (recall that $I = (Q + W) : \mathfrak{m}$ by Lemma (3.5)). Hence $I^2 \subseteq QI + W$, so that we have $I^2 = QI$ since $W \cap Q = (0)$ (cf. (2.6) and (2.7)). $\square$

We now turn to other topics.

Theorem (3.9). *Let A be a Buchsbaum local ring with $d = \dim A \geq 1$ and $\mathrm{e}(A) > 1$. Let Q be a parameter ideal in A and put $I = Q : \mathfrak{m}$. Then $I^2 = QI$ if $\ell_A(I/Q) = \mathrm{r}(A)$.*

Proof. Let $W = \mathrm{H}^0_{\mathfrak{m}}(A)$. Then $\mathfrak{m}W = (0)$ and $Q \subseteq Q + W \subseteq I \subseteq (Q + W) : \mathfrak{m}$. Hence

$$\ell_A(I/Q) = \ell_A(I/(Q + W)) + \ell_A(W)$$

because $W \cap Q = (0)$. Assume that $d = 1$. Then $\mathrm{r}(A) = \ell_A(W) + \mu_{\hat{A}}(\mathrm{K}_{\hat{A}}) = \ell_A(I/Q)$. Since A/W is a Cohen-Macaulay ring and $\mathrm{H}^1_{\mathfrak{m}}(A) \cong \mathrm{H}^1_{\mathfrak{m}}(A/W)$, we have

$$\mu_{\hat{A}}(\mathrm{K}_{\hat{A}}) = \mathrm{r}(A/W) = \ell_A\left([(Q + W) : \mathfrak{m}]/(Q + W)\right)$$

so that

$$\ell_A\left([(Q+W):\mathfrak{m}]/(Q+W)\right) = \mu_{\hat{A}}(\mathrm{K}_{\hat{A}}) = \ell_A(I/Q) - \ell_A(W) = \ell_A(I/(Q+W)).$$

Hence $(Q+W):\mathfrak{m} = I$ and so $I^2 = QI$ (cf. Proof of Corollary (2.11)).

Assume now that $d \geq 2$ and that our assertion holds true for $d-1$. Let $Q = (a_1, a_2, \cdots, a_d)$ and put $\overline{A} = A/(a_1)$, $\overline{Q} = Q/(a_1)$, $\overline{I} = I/(a_1)$, and $\overline{\mathfrak{m}} = \mathfrak{m}/(a_1)$. Then $\overline{I} = \overline{Q} : \overline{\mathfrak{m}}$ and $\mathrm{r}(\overline{A}) \geq \ell_{\overline{A}}(\overline{I}/\overline{Q}) = \ell_A(I/Q) = \mathrm{r}(A)$. Hence by Proposition (3.1) we get $\mathrm{r}(\overline{A}) = \ell_{\overline{A}}(\overline{I}/\overline{Q})$, so that $\overline{I}^2 = \overline{Q}\,\overline{I}$ by the hypothesis on d. Thus $I^2 \subseteq (a_2, , \cdots, a_d)I + (a_1)$ and then the equality $I^2 = QI$ follows similarly as in the proof of Claim (2.12). $\square$

The following is a direct consequence of Theorem (3.9), which may account well for the reason why $I^2 = QI$ in Cohen-Macaulay rings A.

Corollary (3.10). *Let A be a Buchsbaum local ring with $d = \dim A \geq 1$ and assume that the index $\ell_A((Q:\mathfrak{m})/Q)$ of reducibility of Q is independent of the choice of a parameter ideal Q in A. If $\mathrm{e}(A) > 1$, then the equality $I^2 = QI$ holds true for every parameter ideal Q in A, where $I = Q : \mathfrak{m}$.*

The hypothesis of Corollary (3.10) may be satisfied even though A is not a Cohen-Macaulay ring. Let $B = \mathbb{C}[[X,Y,Z]]/(Z^2 - XY)$ where $\mathbb{C}[[X,Y,Z]]$ is the formal power series ring over the field $\mathbb{C}$ of complex numbers, and put

$$A = \mathbb{R}[[x, y, z, ix, iy, iz]]$$

where $\mathbb{R}$ is the field of real numbers, $i = \sqrt{-1}$, and x, y, and z denote the reduction of X, Y, and Z mod $(Z^2 - XY)$. Then A is a Buchsbaum local integral domain with $\dim A = 2$, $\operatorname{depth} A = 1$, and $\mathrm{e}(A) = 4$. For this ring A one has the equality

$$\ell_A((Q:\mathfrak{m})/Q) = 4$$

for every parameter ideal Q in A ([GSu, Example (4.8)]). Hence by Corollary (3.10), $I^2 = QI$ for all parameter ideals Q in A.

The following theorem (3.11) gives an answer to the question raised in the previous section. The authors know no example of Buchsbaum local rings A with $\mathrm{e}(A) > 1$ such that $I^2 \neq QI$ for some parameter ideal $Q \subseteq \mathfrak{m}^2$.

Theorem (3.11). *Let A be a Buchsbaum local ring and assume that $\dim A \geq 2$ or that $\dim A = 1$ and $\mathrm{e}(A) > 1$. Then there exists an integer $\ell \gg 0$ such that $I^2 = QI$ for every parameter ideal $Q \subseteq \mathfrak{m}^\ell$.*

To prove this theorem we need one more lemma. Let A be an arbitrary Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A \geq 1$. Let $f : M \to N$ be a

homomorphism of A-modules. Then we say that f is surjective (resp. bijective) on the socles, if the induced homomorphism

$$f_* : \mathrm{Hom}_A(A/\mathfrak{m}, M) = (0) :_M \mathfrak{m} \to \mathrm{Hom}_A(A/\mathfrak{m}, N) = (0) :_N \mathfrak{m}$$

is an epimorphism (resp. an isomorphism).

Let $Q = (a_1, a_2, \cdots, a_d)$ be a parameter ideal in A and let M be an A-module. For each integer $n \geq 1$ we denote by $\underline{a}^n$ the sequence $a_1^n, a_2^n, \cdots, a_d^n$. Let $\mathrm{K}_\bullet(\underline{a}^n)$ be the Koszul complex of A generated by the sequence $\underline{a}^n$ and let

$$\mathrm{H}^\bullet(\underline{a}^n; M) = H^\bullet(\mathrm{Hom}_A(\mathrm{K}_\bullet(\underline{a}^n), M))$$

be the Koszul cohomology module of M. Then for every $p \in \mathbb{Z}$ the family $\{\mathrm{H}^p(\underline{a}^n; M)\}_{n \geq 1}$ naturally forms an inductive system of A-modules, whose limit

$$\mathrm{H}^p_{\underline{a}}(M) = \lim_{n \to \infty} \mathrm{H}^p(\underline{a}^n; M)$$

is isomorphic to the local cohomology module

$$\mathrm{H}^p_{\mathfrak{m}}(M) = \lim_{n \to \infty} \mathrm{Ext}^p_A(A/\mathfrak{m}^n, M).$$

For each $n \geq 1$ and $p \in \mathbb{Z}$ let $\varphi^{p,n}_{\underline{a},M} : \mathrm{H}^p(\underline{a}^n; M) \to \lim\limits_{n \to \infty} \mathrm{H}^p_{\underline{a}}(M)$ denote the canonical homomorphism into the limit. With this notation we have the following.

Lemma (3.12). *Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A \geq 1$. Let M be a finitely generated A-module. Then there exists an integer $\ell \gg 0$ such that for all systems $a_1, a_2, \cdots, a_d$ of parameters for A contained in $\mathfrak{m}^\ell$ and for all $p \in \mathbb{Z}$ the canonical homomorphisms*

$$\varphi^{p,1}_{\underline{a},M} : \mathrm{H}^p(\underline{a}; M) \to \mathrm{H}^p_{\underline{a}}(M) = \lim_{n \to \infty} \mathrm{H}^p(\underline{a}^n; M)$$

into the inductive limit are surjective on the socles.

Proof. First of all, choose $\ell \gg 0$ so that the canonical homomorphisms

$$\varphi^{p,\ell}_{\mathfrak{m},M} : \mathrm{Ext}^p_A(A/\mathfrak{m}^\ell, M) \to \mathrm{H}^p_{\mathfrak{m}}(M) = \lim_{n \to \infty} \mathrm{Ext}^p_A(A/\mathfrak{m}^n, M)$$

are surjective on the socles for all $p \in \mathbb{Z}$. This choice is possible, because $\mathrm{H}^p_{\mathfrak{m}}(M) = (0)$ for almost all $p \in \mathbb{Z}$ and the socle of $[(0) :_{\mathrm{H}^p_{\mathfrak{m}}(M)} \mathfrak{m}]$ of $\mathrm{H}^p_{\mathfrak{m}}(M)$ is finitely generated. Let $Q = (a_1, a_2, \cdots, a_d)$ be a parameter ideal in A and assume that $Q \subseteq \mathfrak{m}^\ell$. Then, since $\sqrt{Q} =$

$\sqrt{\mathfrak{m}^\ell} = \mathfrak{m}$, there exists an isomorphism $\theta_M^p : \mathrm{H}^p_{\mathfrak{m}}(M) \to \mathrm{H}^p_Q(M) = \lim\limits_{n\to\infty} \mathrm{Ext}^p_A(A/Q^n, M)$ which makes the diagram

$$\begin{array}{ccc}
\mathrm{Ext}^p_A(A/\mathfrak{m}^\ell, M) & \xrightarrow{\varphi^{p,\ell}_{\mathfrak{m},M}} & \mathrm{H}^p_{\mathfrak{m}}(M) \\
\alpha \big\downarrow & & \big\downarrow \theta^p_M \\
\mathrm{Ext}^p_A(A/Q, M) & \xrightarrow[\varphi^{p,1}_{Q,M}]{} & \mathrm{H}^p_Q(M)
\end{array}$$

commutative, where the vertical map $\alpha : \mathrm{Ext}^p_A(A/\mathfrak{m}^\ell, M) \to \mathrm{Ext}^p_A(A/Q, M)$ is the homomorphism induced from the epimorphism $A/Q \to A/\mathfrak{m}^\ell$. Hence the homomorphism $\varphi^{p,1}_{Q,M}$ is surjective on the socles, since so is $\varphi^{p,\ell}_{\mathfrak{m},M}$. Let $n \geq 1$ be an integer and let

$$\cdots \to F_i \to \cdots \to F_1 \to F_0 = A \to A/Q^n \to 0$$

be a minimal free resolution of A/Q^n. Then since $(\underline{a}^n) \subseteq Q^n$, the epimorphism

$$\varepsilon : A/(\underline{a}^n) \to A/Q^n$$

can be lifted to a homomorphism of complexes:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & F_i & \longrightarrow & \cdots & \longrightarrow & F_1 & \longrightarrow & F_0 = A & \longrightarrow & A/Q^n & \longrightarrow & 0 \\
 & & \big\uparrow & & & & \big\uparrow & & \big\| & & \big\uparrow \varepsilon & & \\
\cdots & \longrightarrow & K_i & \longrightarrow & \cdots & \longrightarrow & K_1 & \longrightarrow & K_0 = A & \longrightarrow & A/(\underline{a}^n) & \longrightarrow & 0
\end{array}$$

where $K_\bullet = \mathrm{K}_\bullet(\underline{a}^n)$. Taking the M-dual of these two complexes and passing to the cohomology modules, we get the natural homomorphism

$$\alpha^{p,n}_M : \mathrm{Ext}^p_A(A/Q^n, M) \to \mathrm{H}(\underline{a}^n; M)$$

$(p \in \mathbb{Z}, n \geq 1)$ of inductive systems, whose limit

$$\alpha^p_M : \mathrm{H}^p_Q(M) \to \mathrm{H}^p_{\underline{a}}(M)$$

is necessarily an isomorphism for all $p \in \mathbb{Z}$. Consequently, thanks to the commutative diagram

$$\begin{array}{ccc}
\mathrm{Ext}^p_A(A/Q, M) & \xrightarrow{\varphi^{p,1}_{Q,M}} & \mathrm{H}^p_Q(M) \\
\alpha^{p,1}_M \big\downarrow & & \big\downarrow \alpha^p_M \\
\mathrm{H}^p(\underline{a}; M) & \xrightarrow[\varphi^{p,1}_{\underline{a},M}]{} & \mathrm{H}^p_{\underline{a}}(M)
\end{array}$$

we get that for all $p \in \mathbb{Z}$ the homomorphism

$$\varphi^{p,1}_{\underline{a},M} : \mathrm{H}^p(\underline{a}; M) \to \mathrm{H}^p_{\underline{a}}(M)$$

is surjective on the socles, because so is $\varphi^{p,1}_{Q,M}$. $\square$

Corollary (3.13). *Let A be a Buchsbaum local ring with $d = \dim A \geq 1$. Then there exists an integer $\ell \gg 0$ such that the index $\ell_A((Q : \mathfrak{m})/Q)$ of reducibility of Q is independent of Q and equals $\mathrm{r}(A)$ for all parameter ideals $Q \subseteq \mathfrak{m}^\ell$.*

Proof. Choose an integer $\ell \gg 0$ so that the canonical homomorphism

$$\varphi_{\underline{a},A}^{d,1} : A/Q = \mathrm{H}^d(\underline{a}; A) \to \mathrm{H}_{\underline{a}}^d(A)$$

is surjective on the socles for every parameter ideal $Q = (a_1, a_2, \cdots, a_d) \subseteq \mathfrak{m}^\ell$. Then since A is a Buchsbaum local ring, we get that

$$\mathrm{Ker}\ \varphi_{\underline{a},A}^{d,1} = \sum_{i=1}^{d} \left[\left((a_1, \cdots, \overset{\vee}{a_i}, \cdots, a_d) : a_i\right) + Q\right]/Q$$

([G2, Theorem (4.7)]), $\mathfrak{m}{\cdot}[\mathrm{Ker}\ \varphi_{\underline{a},A}^{d,1}] = (0)$, and $\ell_A(\mathrm{Ker}\ \varphi_{\underline{a},A}^{d,1}) = \sum_{i=0}^{d-1} \binom{d}{i} h^i(A)$ ([G2, Proposition (3.6)]). Because $\mu_{\hat{A}}(\mathrm{K}_{\hat{A}}) = \ell_A((0) :_{\mathrm{H}_{\underline{a}}^d(A)} \mathfrak{m})$, the surjectivity of the homomorphism $\varphi_{\underline{a},A}^{d,1}$ on the socles guarantees that

$$\ell_A(I/Q) = \sum_{i=0}^{d-1} \binom{d}{i} h^i(A) + \mu_{\hat{A}}(\mathrm{K}_{\hat{A}})$$

where $I = Q : \mathfrak{m}$. Hence $\mathrm{r}(A) = \ell_A(I/Q)$. $\square$

We are now ready to prove Theorem (3.11).

Proof of Theorem (3.11). Thanks to Theorem (3.9) and Corollary (3.13) we may assume that $\mathrm{e}(A) = 1$ and $d \geq 2$. Let $W = \mathrm{H}_{\mathfrak{m}}^0(A)$ and $B = A/W$. Then B is a regular local ring with $d = \dim B \geq 2$. We choose a parameter ideal Q in A so that $Q \subseteq \mathfrak{m}^2$. Let $J = QB : \mathfrak{m}B$. Then since $QB \subseteq (\mathfrak{m}B)^2$, by Theorem (1.1) we get $J^2 = QB{\cdot}J$. Because B/QB is a Gorenstein ring and $QB \subseteq IB \subseteq J$, we have either $IB = QB$ or $IB = J$. In any case $I^2 \subseteq QI + W$, so that $I^2 = QI$, because $W \cap Q = (0)$. $\square$

4. Evaluation of $\mathrm{r}_Q(I)$ in the case where $\dim A = 1$.

In this section let A be a Buchsbaum local ring and assume that $\dim A = 1$. Let $W = \mathrm{H}_{\mathfrak{m}}^0(A)$ $(= (0) : \mathfrak{m})$ and $e = \mathrm{e}(A)$. Then $\mathrm{r}(A) = \ell_A(W) + \mathrm{r}(A/W)$ and $\mathrm{r}(A/W) \leq \max\{1, e-1\}$, since A/W is a Cohen-Macaulay local ring with $\mathrm{e}(A/W) = e$ (cf. [HK, Bemerkung 1.21 b)]). The purpose is to prove the following.

Theorem (4.1). *Suppose that $e > 1$. Let Q be a parameter ideal in A and put $I = Q : \mathfrak{m}$. Then*

$$\mathrm{r}_Q(I) \leq \mathrm{r}(A) - \ell_A(W) + 1 = \mathrm{r}(A/W) - \ell_A(I/(Q+W)) + 1.$$

Proof. Let $Q = (a)$ and put $I_n = I^{n+1} : a^n$ $(n \geq 0)$. Then $I_0 = I$ and $I_n \subseteq I_{n+1}$. We have $I_n \subseteq (Q + W) : \mathfrak{m}$. In fact, let $x \in I_n$ and $\alpha \in \mathfrak{m}$. Then $a^n(\alpha x) \in \mathfrak{m}I^{n+1} \subseteq (a^{n+1})$ by Proposition (2.3). Let $a^n(\alpha x) = a^{n+1}y$ with $y \in A$. Then $\alpha x - ay \in (0) : a^n = W$, whence $x \in (Q + W) : \mathfrak{m}$. We furthermore have the following.

Claim (4.2). *Let $n \geq 0$ and assume that $I_n = I_{n+1}$. Then $I^{n+2} = QI^{n+1}$.*

Proof of Claim (4.2). Let $x \in I^{n+2} \subseteq (a^{n+1})$ and write $x = a^{n+1}y$ with $y \in A$. Then $y \in I^{n+2} : a^{n+1} = I_n$, so that $x = a(a^n y) \in QI^{n+1}$. Thus $I^{n+2} = QI^{n+1}$. $\square$

Let $\ell = \ell_A(I/(Q + W))$. Then $\mathrm{r}(A/W) = \ell_A([(Q + W) : \mathfrak{m}]/(Q + W)) \geq \ell$. Since $\ell_A(I/Q) = \ell_A(I/(Q + W)) + \ell_A(W)$ (cf. Proof of Theorem (3.9)), we get

$$\begin{aligned} \mathrm{r}(A) - \ell_A(I/Q) + 1 &= [\mathrm{r}(A/W) + \ell_A(W)] - [\ell_A(I/(Q + W)) + \ell_A(W)] + 1 \\ &= \mathrm{r}(A/W) - \ell_A(I/(Q + W)) + 1 \\ &= \mathrm{r}(A/W) - \ell + 1. \end{aligned}$$

Assume that $\mathrm{r}_Q(I) > \mathrm{r}(A/W) - \ell + 1$ and put $n = \mathrm{r}(A/W) - \ell + 2$. Then $\mathrm{r}_Q(I) \geq n \geq 2$, so that by Claim (4.2) $I_i \neq I_{i+1}$ for all $0 \leq i \leq n - 2$. Hence we have a chain

$$Q + W \subseteq I_0 = I \subsetneq I_1 \subsetneq \cdots \subsetneq I_{n-2} \subsetneq I_{n-1} \subseteq (Q + W) : \mathfrak{m}$$

of ideals, so that $\mathrm{r}(A/W) = \ell_A([(Q + W) : \mathfrak{m}]/(Q + W)) \geq (n - 1) + \ell = \mathrm{r}(A/W) + 1$, which is absurd. Thus $\mathrm{r}_Q(I) \leq \mathrm{r}(A/W) - \ell + 1$. $\square$

Suppose that $e > 1$ and let Q be a parameter ideal in A. Let $I = Q : \mathfrak{m}$. Then $I \supseteq Q + W$. We have by Theorem (4.1) that $\mathrm{r}_Q(I) \leq \mathrm{r}(A/W) \leq e - 1$, if $I \supsetneq Q + W$. If $I = Q + W$, then $I^2 = Q^2$ because $\mathfrak{m}W = (0)$, so that $I^n = Q^n$ for all $n \geq 2$. Thus we have

Corollary (4.3). *Let A be a Buchsbaum local ring with $\dim A = 1$ and $e = \mathrm{e}(A) > 1$. Then*

$$\sup_Q \ \mathrm{r}_Q(Q : \mathfrak{m}) \leq e - 1$$

where Q runs over parameter ideals in A.

The evaluations in Theorem (4.1) and Corollary (4.3) are sharp, as we shall show in the following example. The example shows that for every integer $e \geq 3$ there exists a Buchsbaum local ring A with $\dim A = 1$ and $\mathrm{e}(A) = e$ which contains a parameter ideal Q such that $\mathrm{r}_Q(I) = e - 1$, where $I = Q : \mathfrak{m}$. Hence the equality $I^2 = QI$ fails in general to hold, even though A is a Buchsbaum local ring with $\mathrm{e}(A) > 1$. The reader may consult the forthcoming paper [GSa] for higher-dimensional examples of higher depth.

Let k be a field and $3 \leq e \in \mathbb{Z}$. Let $S = k[X_1, X_2, \cdots, X_e]$ and $P = k[t]$ be the polynomial rings over k. We regard S and P as $\mathbb{Z}$-graded rings whose gradings are given by $S_0 = k$, $S_{e+i-1} \ni X_i$ $(1 \leq i \leq e)$ and $P_0 = k$, $P_1 \ni t$. Hence $S_n = (0)$ for $1 \leq n \leq e$, where S_n denotes the homogeneous component of S with degree n. Let $\varphi : S \to P$ be the k-algebra map defined by $\varphi(X_i) = t^{e+i-1}$ for all $1 \leq i \leq e$. Then φ is a homomorphism of graded rings, whose image is the semigroup ring $k[t^e, t^{e+1}, \cdots, t^{2e-1}]$, and whose kernel $\mathfrak{p}$ is minimally generated by the 2 by 2 minors of the matrix

$$\mathbb{M} = \begin{pmatrix} X_1 & X_2 & \cdots & X_{e-1} & X_e \\ X_2 & X_3 & \cdots & X_e & X_1^2 \end{pmatrix}.$$

Let Δ_{ij} $(1 \leq i, j \leq e)$ be the determinant of the matrix consisting of the i $\underline{th}$ and j $\underline{th}$ columns of $\mathbb{M}$, that is

$$\Delta_{ij} = \begin{vmatrix} X_i & X_j \\ X_{i+1} & X_{j+1} \end{vmatrix},$$

where $X_{e+1} = X_1^2$ for convention. We put $\Delta = \Delta_{2,e}$ and let $N = S_+$ $(= \bigoplus_{n \geq 1} S_n)$, the unique graded maximal ideal in S. Let

$$\mathfrak{a} = (\Delta_{ij} \mid 1 \leq i < j \leq e \ \text{ such that } \ (i,j) \neq (2,e)) + \Delta N$$

and put $R = S/\mathfrak{a}$, $M = R_+$, $A = R_M$, and $\mathfrak{m} = MA$. Let $x_i = X_i \bmod \mathfrak{a}$ $(1 \leq i \leq e)$ and $\delta = \Delta \bmod \mathfrak{a}$. We then have the following.

Lemma (4.4). *$\dim R = 1$, $\mathrm{H}^0_M(R) = (\delta) \neq (0)$, and $M\delta = (0)$.*

Proof. We certainly have $M\delta = (0)$. Look at the canonical exact sequence

(4.5) $$0 \to \mathfrak{p}/\mathfrak{a} = (\delta) \to R \to S/\mathfrak{p} \to 0,$$

where $\mathfrak{p} = \operatorname{Ker} \varphi$. Then, since $M\delta = (0)$ and $S/\mathfrak{p} = k[t^e, t^{e+1}, \cdots, t^{2e-1}]$ is a Cohen-Macaulay integral domain with $\dim S/\mathfrak{p} = 1$, we get that $\dim R = 1$ and $\mathrm{H}^0_M(R) = (\delta)$. The assertion $\delta \neq 0$ follows from the fact that $\{\Delta_{ij}\}_{1 \leq i < j \leq e}$ is a *minimal* system of generators for the ideal $\mathfrak{p}$. $\square$

Let $T = k[t^e, t^{e+1}, \cdots, t^{2e-1}]$ and $\mathfrak{n} = T_+$. Then $\mathfrak{n} = (t^e, t^{e+1}, \cdots, t^{2e-1})T$ and $\mathfrak{n}^2 = t^e\mathfrak{n}$. Hence

$$\mathrm{r}(T_\mathfrak{n}) = \ell_T((t^eT : \mathfrak{n})/t^eT) = \ell_T(\mathfrak{n}/t^eT) = e - 1.$$

We have $M^2 = x_1M + (\delta)$, because $\mathfrak{n}^2 = t^e\mathfrak{n}$ and $\delta \in M^2$. Hence $M^3 = x_1M^2$, so that $\mathrm{e}(A) = \mathrm{e}^0_{x_1A}(A) = \mathrm{e}^0_{x_1A}(T_\mathfrak{n}) = \ell_T(T/t^eT) = e$ (cf. (4.5)). Thus A is a Buchsbaum ring with $\dim A = 1$ and $\mathrm{e}(A) = \mathrm{r}(A) = e$. In particular, $\delta \notin (x_1)$, since $(x_1) \cap \mathrm{H}^0_M(R) = (0)$ (recall that x_1 is a parameter of R).

We put $J = (x_1) : M$.

Proposition (4.6). *The following assertions hold true.*

(1) $J = (x_1, x_2, \delta)$.
(2) $J^n = (x_1, x_2)^n$ *for all* $n \geq 2$.
(3) $\ell_R(J/(x_1)) = 2$.

Proof. We firstly notice that

$$
\begin{aligned}
(4.7) \qquad \mathfrak{a} + X_1 \supseteq\ & (X_1) + (X_2, X_3X_e)(X_2, \cdots, X_e) \\
& + (\Delta_{ij} \mid 3 \leq i, j \leq e,\ i + j = e + 2) \\
& + (X_iX_j \mid 3 \leq i, j \leq e,\ i + j \neq e + 3).
\end{aligned}
$$

In fact, $\Delta \equiv -X_3X_e \bmod (X_1)$ and $\Delta_{1,j} = X_1X_{j+1} - X_2X_j \equiv -X_2X_j \bmod (X_1)$, we get $\mathfrak{a} + (X_1) \supseteq (X_1) + (X_2, X_3X_e)(X_2, \cdots, X_e)$. Let $3 \leq i, j \leq e$. If $i + j = e + 2$, then $(i, j) \neq (2, e)$ and $(j, i) \neq (2, e)$, so that $\Delta_{ij} \in \mathfrak{a}$. Assume that $i + j \neq e + 3$. We will show $X_iX_j \in \mathfrak{a} + (X_1)$ by induction on i. If $i = 3$, then $3 \leq j < e$ and $\Delta_{2j} = X_2X_{j+1} - X_3X_j \in \mathfrak{a}$, whence $X_3X_j \in \mathfrak{a} + (X_1)$, because $X_2X_{j+1} \in \mathfrak{a} + (X_1)$. Assume that $i \geq 4$ and that our assertion holds true for $i - 1$. Then $3 \leq i - 1 < e$, so that $\Delta_{i-1,j} = X_{i-1}X_{j+1} - X_iX_j \in \mathfrak{a}$. Hence $X_iX_j \in \mathfrak{a} + (X_1)$, because $X_{i-1}X_{j+1} \in \mathfrak{a} + (X_1)$ by the hypothesis on i.

Let $B = S/(\mathfrak{a} + (X_1))$ and $\mathfrak{q} = B_+$. Then $(B, \mathfrak{q})$ is an Artinian graded local ring. For the moment, let us denote by y_i the reduction of $X_i \bmod \mathfrak{a} + (X_1)$ $(2 \leq i \leq e)$ and by ρ the reduction of $-\Delta \bmod \mathfrak{a} + (X_1)$. Hence $\mathfrak{q} = (y_2, \cdots, y_e)$ and $\rho = y_3y_e$. We will check that $\mathfrak{q}^2 = (\rho)$. To see this, let $2 \leq i, j \leq e$ and assume that $y_iy_j \neq 0$. Then $3 \leq i, j \leq e$ and $i + j = e + 3$ by (4.7), whence $y_iy_j = \rho$, because $\rho = y_3y_e$ and $y_\alpha y_{\beta+1} = y_{\alpha+1}y_\beta$ whenever $3 \leq \alpha, \beta \leq e$ with $\alpha + \beta = e + 3$. Hence $\mathfrak{q}^2 = (\rho)$, so that $\mathfrak{q}^3 = (0)$ because $N{\cdot}\Delta \subseteq \mathfrak{a}$. We have $\rho \neq 0$, since $\Delta \notin \mathfrak{a} + (X_1)$ (recall that $\delta \notin (x_1)$). Now let $\varphi \in (0) : \mathfrak{q}$ and write $\varphi = c + \sum_{i=2}^{e} c_iy_i + d\rho$ with $c, c_i, d \in k$. Then because $(0) : \mathfrak{q}$ is a graded ideal in B and $c_iy_i \in B_{e+i-1}$ for $2 \leq i \leq e$ and $\rho \in B_{3e+1}$, we get $c, c_iy_i, d\rho \in (0) : \mathfrak{q}$. Hence $c = 0$, because $(0) : \mathfrak{q} \subseteq \mathfrak{q}$. We have $c_i = 0$ for all $3 \leq i \leq e$, because $\rho = y_\alpha y_{e-\alpha+3} \neq 0$ for all $3 \leq \alpha \leq e$. Thus $\varphi = c_2y_2 + d\rho \in (y_2, \rho)$. Hence $(0) : \mathfrak{q} = (y_2, \rho)$ by (4.7), so that we have $J = (x_1, x_2, \delta)$ in R. Assrtions (2) and (3) are now clear. $\square$

Theorem (4.8). $J^e = x_1J^{e-1}$ *but* $J^{e-1} \neq x_1J^{e-2}$.

Proof. Assume that $J^{e-1} = x_1J^{e-2}$. Then $J^{e-1} \ni x_2^{e-1} = x_2^2x_2^{e-3} = x_1{\cdot}x_2^{e-3}x_3$. Let $x_1{\cdot}x_2^{e-3}x_3 = x_1\eta$ with $\eta \in J^{e-2}$. Then $x_2^{e-3}x_3 - \eta \in (0) : x_1 = (\delta)$. We write

$$x_2^{e-3}x_3 = \eta + \delta\xi$$

with $\xi \in R$. If $e = 3$, then $x_3 \in J = (x_1, x_2, \delta) \subseteq (x_1, x_3^2)$, which is impossible. Hence $e \geq 4$ and so $\eta \in (x_1)$, since $\eta \in J^{e-2} \subseteq J^2$ and $J^2 = (x_1, x_2)^2 = (x_1^2, x_1x_2, x_2^2) \subseteq (x_1)$

(cf. Proposition (4.2) (2); recall that $x_2^2 = x_1x_3$). Hence $\delta\xi \in (x_1) \cap \mathrm{H}^0_M(R) = (0)$, because $x_2^{e-3}x_3 = x_2x_3{\cdot}x_2^{e-4} = x_1x_4x_2^{e-4} \in (x_1)$. Thus by Proposition (4.2) (2)

$$x_2^{e-3}x_3 = \eta \in (x_1, x_2)^{e-2} = (x_1^ix_2^{e-2-i} \mid 0 \le i \le e-2). \tag{4.9}$$

Here we notice that $R = \bigoplus_{n \ge 0} R_n$ is a graded ring and that $\deg\,(x_1^ix_2^{e-2-i}) = e^2-e-i-2$, $\deg\,(x_2^{e-3}x_3) = e^2-e-1$. Then, since $1 \le i+1 = (e^2-e-1)-(e^2-e-i-2) \le e-1$ for $0 \le i \le e-2$ and $R_n = (0)$ for $1 \le n \le e-1$, by (4.9) we get $x_2^{e-3}x_3 = 0$, whence $X_2^{e-3}X_3 \in \mathfrak{p} = \operatorname{Ker}\varphi$, which is impossible. Thus $J^{e-1} \ne x_1J^{e-2}$. Since $J^e = x_1J^{e-1} + (x_2^e)$, the equality $J^e = x_1J^{e-1}$ follows from Corollary (4.3), or more directly from the following.

Claim (4.10). $x_2^e = x_1^{e+1}$.

Proof of Claim (4.10). It suffices to show $x_2^e = x_1^nx_2^{e-n-1}x_{n+2}$ for all $1 \le n \le e-2$. Since $x_2^e = x_1x_3{\cdot}x_2^{e-2}$, the assertion is obviously true for $n = 1$. Let $n \ge 2$ and assume that the equality holds true for $n-1$. Then

$$\begin{aligned} x_2^e &= x_1^{n-1}x_2^{e-n}x_{n+1} \\ &= x_1^{n-1}x_2^{e-n-1}{\cdot}x_2x_{n+1} \\ &= x_1^nx_2^{e-n-1}x_{n+2}, \end{aligned}$$

because $x_2x_{n+1} = x_1x_{n+2}$. Hence $x_2^e = x_1^{e-2}{\cdot}x_2x_e = x_1^{e-2}x_1^3 = x_1^{e+1}$. $\square$

Let $Q = x_1A$ and $I = Q : \mathfrak{m}$ $(= JA)$. Then in our Buchsbaum local ring A we have $I^e = x_1I^{e-1}$ but $I^{e-1} \ne x_1I^{e-2}$. Because $\mathrm{e}(A) = \mathrm{r}(A) = e$, this example shows the evaluations in Theorem (4.1) and Corollary (4.3) are really sharp.

5. Examples

In this section we shall explore two examples. One is to show that the equality $I^2 = QI$ may hold true for *all* parameter ideals Q in A, even though A is not a generalized Cohen-Macaulay ring. As is shown in the previous section, the equality $I^2 = QI$ fails in general to hold, even though A is a Buchsbaum local ring with $\mathrm{e}(A) > 1$. In this section we will also explore one counterexample of dimension 1 and give complete criteria of the equality $I^2 = QI$ for parameter ideals Q in the example.

Throughout this section let $(R, \mathfrak{n})$ be a 3-dimensional regular local ring and let $\mathfrak{n} = (X, Y, Z)$. Firstly, let $\ell \ge 1$ be an integer and put

$$A = R/(X^\ell) \cap (Y, Z).$$

Let x, y, and z denote the reduction of X, Y, and Z mod $(X^\ell) \cap (Y, Z) = (X^\ell Y, X^\ell Z)$. Let $\mathfrak{p} = (y, z)$. Then $\mathfrak{m} = (x) + \mathfrak{p}$ and $(x^\ell) \cap \mathfrak{p} = (0)$ in A, where $\mathfrak{m}$ denotes the maximal ideal in A. Let $B = A/(x^\ell)$. Then there exists exact sequences

$$0 \to A/\mathfrak{p} \xrightarrow{\alpha} A \to B \to 0 \quad \text{and} \tag{5.1}$$

$$0 \to A/(x) \xrightarrow{\beta} B \to A/(x^{\ell-1}) \to 0 \tag{5.2}$$

of A-modules, where the homomorphisms α and β are defined by $\alpha(1) = x^\ell$ and $\beta(1) = x^{\ell-1}$ mod (x^ℓ). Since $A/\mathfrak{p}$ is a DVR and B is a hypersurface with $\dim B = 2$, we get by (5.1) that

$$\dim A = 2, \ \operatorname{depth} A = 1, \ \text{and } \mathrm{H}^1_{\mathfrak{m}}(A/\mathfrak{p}) \cong \mathrm{H}^1_{\mathfrak{m}}(A).$$

Hence A is not a generalized Cohen-Macaulay ring. Let $\mathfrak{q} = (x-y, z)$. Then $\mathfrak{m}^{\ell+1} = \mathfrak{q}\mathfrak{m}^\ell$, since $\mathfrak{m} = (x) + \mathfrak{q}$ and $x^{\ell+1} = (x-y)x^\ell$. Consequently by (5.1) we get

$$\mathrm{e}(A) = \mathrm{e}^0_{\mathfrak{q}}(A) = \mathrm{e}^0_{\mathfrak{q}}(B) = \ell_A(B/\mathfrak{q}B) = \ell_R(R/(X^\ell, X-Y, Z)).$$

Hence $\mathrm{e}(A) = \ell$. We furthermore have the following.

Theorem (5.3). *Let Q be a parameter ideal in A and $I = Q : \mathfrak{m}$. Then $\ell_A(I/Q) \le 2$. The equality $I^2 = QI$ holds true if and only if one of the following conditions is satisfied.*

(1) $\ell \ge 2$.
(2) $\ell = 1$ *and* $\ell_A(I/Q) = 1$.
(3) $\ell = 1$, $\ell_A(I/Q) = 2$, *and* $QB \ne (QB)^\sharp$ *in* $B = A/(x)$.

Hence $I^2 = QI$ if either $\ell \ge 2$, or $\ell = 1$ and $Q \subseteq \mathfrak{m}^2$.

Proof. Let $Q = (f, g)$. Then the sequence f, g is B-regular, so that by (5.1) we get the exact sequence

$$0 \to A/(\mathfrak{p} + Q) \to A/Q \to B/QB \to 0. \tag{5.4}$$

Hence $\ell_A(I/Q) \le 2$, because both the rings $A/(\mathfrak{p}+Q)$ and B/QB are Gorenstein. Since $A/\mathfrak{p}$ is a DVR and $(Q+\mathfrak{p})/\mathfrak{p} = (\overline{f}, \overline{g})$, we may assume that $(Q+\mathfrak{p})/\mathfrak{p} = (\overline{f}) \ni \overline{g}$ (here $\overline{*}$ denotes the reduction mod $\mathfrak{p}$). Let $\overline{g} = \overline{cf}$ with $c \in A$. Then, since $Q = (f, g - cf)$, replacing g by $g - cf$, we get $Q = (f, g)$ with $g \in \mathfrak{p}$. Since $\mathfrak{m}/\mathfrak{p} = (\overline{x})$, letting $\overline{f} = \overline{\varepsilon}\, \overline{x}^n$ with $\varepsilon \in \mathrm{U}(A)$ and $n \ge 1$, we have $Q = (\varepsilon x^n + a_1, g)$ for some $a_1 \in \mathfrak{p}$. Hence $Q = (x^n + \varepsilon^{-1} a_1, g)$, so that

$$Q = (x^n + a, b) \tag{5.5}$$

with $a, b \in \mathfrak{p}$ and $n \geq 1$. We then have by (5.4) the exact sequence

$$0 \to A/((x^n) + \mathfrak{p}) \xrightarrow{\gamma} A/Q \to B/QB \to 0, \tag{5.6}$$

where $\gamma(1) = x^\ell \bmod Q$. We notice that $A/((x^n)+\mathfrak{p}) = R/(X^n, Y, Z)$ is a Gorenstein ring, containing $x^{n-1} \bmod (x^n)+\mathfrak{p}$ as the non-zero socle. Then by (5.6) $\gamma(x^{n-1} \bmod (x^n)+\mathfrak{p}) = x^{n+\ell-1} \bmod Q$ is a non-zero element of I/Q, that is

$$Q + (x^{n+\ell-1}) \subseteq I \quad \text{and} \quad x^{n+\ell-1} \notin Q. \tag{5.7}$$

Because $x^{n+\ell-1}a = 0$ (since $x^\ell \mathfrak{p} = (0)$), we get $(x^{n+\ell-1})^2 = (x^n + a)x^{n+\ell-1}x^{\ell-1}$. Hence $(x^{n+\ell-1})^2 \in QI$. This guarantees that $I^2 = QI$ when $\ell_A(I/Q) = 1$, because $I = Q + (x^{n+\ell-1})$ by (5.7).

Now assume that $\ell_A(I/Q) = 2$ and $\mathrm{e}(A) = \ell \geq 2$. Then $\mathfrak{m}I = \mathfrak{m}Q$ by Proposition (2.3), whence

$$\mu_A(I) = \ell_A(I/\mathfrak{m}I) = \ell_A(I/\mathfrak{m}Q) = \ell_A(I/Q) + \ell_A(Q/\mathfrak{m}Q) = 4, \tag{5.8}$$

so that $Q + (x^{n+\ell-1}) \subsetneq I$. Let $I = Q + (x^{n+\ell-1}) + (\xi)$ with $\xi \in A$. Then, since B/QB is a Gorenstein ring and the canonical epimorphism $A/Q \to B/QB$ in (5.6) is surjective on the socles, we have $IB = QB + \xi B = QB : \mathfrak{m}B$. Look at the exact sequence

$$0 \to A/((x) + Q) \xrightarrow{\delta} B/QB \to A/((x^{\ell-1}) + Q) \to 0 \tag{5.9}$$

induced from (5.2), where $\delta(1) = x^{\ell-1} \bmod QB$. Then since $A/((x) + Q)$ is an Artinian Gorenstein ring, choosing $\Delta \in A$ so that $\mathfrak{m}\Delta \subseteq (x) + Q$ but $\Delta \notin (x) + Q$, by (5.9) we have that $x^{\ell-1}\Delta \notin QB$ and

$$IB = QB : \mathfrak{m}B = QB + x^{\ell-1}\Delta B = QB + \xi B.$$

Let us write $\xi = \varepsilon x^{\ell-1}\Delta + \rho_0 + x^\ell \varphi_0$ with $\varepsilon \in \mathrm{U}(A)$, $\rho_0 \in Q$, and $\varphi_0 \in A$. Then $I = Q + (x^{n+\ell-1}) + (\xi) = Q + (x^{n+\ell-1}) + (x^{\ell-1}\Delta + \rho + x^\ell\varphi)$, where $\rho = \varepsilon^{-1}\rho_0$ and $\varphi = \varepsilon^{-1}\varphi_0$. Hence

$$I = Q + (x^{n+\ell-1}) + (x^{\ell-1}\Delta + x^\ell\varphi)$$

because $\rho \in Q$. We need the following.

Claim (5.10). $\Delta \in \mathfrak{m} = (x) + \mathfrak{p}$.

Proof of Claim (5.10). Assume $\Delta \notin \mathfrak{m}$. Then since $x^{\ell-1}(\Delta + x\varphi) \in I$, we have $x^{\ell-1} \in I$, so that $I = Q + (x^{\ell-1})$. This is impossible, because $\mu_A(I) = 4$ by (5.8). $\square$

We write $\Delta = x\sigma + \tau$ with $\sigma \in A$ and $\tau \in \mathfrak{p}$. Then $x^{\ell-1}\Delta + x^\ell\varphi = x^{\ell-1}\tau + x^\ell(\sigma + \varphi)$ and so

$$I = Q + (x^{n+\ell-1}) + (x^{\ell-1}\tau + x^\ell\varphi_1) \tag{5.11}$$

where $\varphi_1 = \sigma + \varphi$. Suppose that $\varphi_1 \notin \mathfrak{p}$ and write $\varphi_1 = \varepsilon_1 x^q + \psi_1$ with $\varepsilon_1 \in \mathrm{U}(A)$, $q \geq 1$, and $\psi_1 \in \mathfrak{p}$. Then $x^{\ell-1}\tau + x^\ell\varphi_1 = x^{\ell-1}\tau + \varepsilon_1 x^{q+\ell}$ because $x^\ell\mathfrak{p} = (0)$. Therefore, letting $\tau_1 = \varepsilon_1^{-1}\tau$, we get

$$I = Q + (x^{n+\ell-1}) + (x^{\ell-1}\tau_1 + x^{q+\ell}).$$

Because $x^\ell\tau_1 = 0$, we have $x^{q+\ell+1} = x(x^{\ell-1}\tau_1 + x^{q+\ell})$, so that $q + \ell + 1 > n + \ell - 1$ since $\mu_A(I) = 4$ (otherwise, $I = Q + (x^{\ell-1}\tau_1 + x^{q+\ell})$). Consequently $x^{q+\ell} = x^{n+\ell-1}(x^{(q+\ell)-(n+\ell-1)})$ and so $I = Q + (x^{n+\ell-1}) + (x^{\ell-1}\tau_1)$ with $\tau_1 \in \mathfrak{p}$. Thus in the expression (5.11) of I we may assume that $\varphi_1 \in \mathfrak{p}$, whence

$$I = Q + (x^{n+\ell-1}) + (x^{\ell-1}\tau)$$

with $\tau \in \mathfrak{p}$. Therefore $I^2 = QI + (x^{n+\ell-1}, x^{\ell-1}\tau)^2 = QI$, because $(x^{n+\ell-1})^2 \in QI$ by (5.7) and $x^{\ell-1}\tau(x^{n+\ell-1}, x^{\ell-1}\tau) = (0)$ (since $x^\ell\mathfrak{p} = (0)$). Thus $I^2 = QI$, if $\ell \geq 2$ or if $\ell = 1$ and $\ell_A(I/Q) = 1$.

We now consider the case where $\mathrm{e}(A) = \ell = 1$ and $\ell_A(I/Q) = 2$. Our ideal I has in this case the following normal form

$$I = Q + (x^n, \xi)$$

where $\xi \in \mathfrak{p}$. In fact, $Q + (x^n) \subseteq I$ and $x^n \notin Q$ by (5.7). Since $\ell_A(I/Q) = 2$, the canonical epimorphism $A/Q \to B/QB$ in (5.6) is surjective on the socles. Hence $IB = QB : \mathfrak{m}B \supsetneq QB$. Let $I = Q + (x^n) + (\xi)$ with $\xi \in A$. If $\xi \notin \mathfrak{p}$, letting $\xi = \varepsilon x^q + \xi_1$ with $\varepsilon \in \mathrm{U}(A)$, $q \geq 1$, and $\xi_1 \in \mathfrak{p}$, we get $x\xi = \varepsilon x^{q+1} \in Q$ (recall that $x\mathfrak{p} = (0)$, since $\ell = 1$). Hence $x^{q+1} \in Q$, so that $\overline{x}^{q+1} \in (\overline{x}^n) = (Q + \mathfrak{p})/\mathfrak{p}$ in the DVR $A/\mathfrak{p}$ (cf. (5.5)). Thus $q + 1 \geq n$. If $q + 1 = n$, then $x^n \in Q$, which is impossible by (5.7). Hence $q \geq n$, and so

$$I = Q + (x^n) + (\varepsilon x^q + \xi_1) = Q + (x^n, \xi_1)$$

with $\xi_1 \in \mathfrak{p}$. Thus, replacing ξ by ξ_1 in the case where $\xi \notin \mathfrak{p}$, we get

$$I = Q + (x^n, \xi) = (x^n, a, b, \xi) \tag{5.12}$$

with $a, b, \xi \in \mathfrak{p}$. If $QB \neq (QB)^\sharp$ in the regular local ring $B = A/(x)$, we have $(IB)^2 = QB{\cdot}IB$ by Theorem (1.1), since $IB = QB : \mathfrak{m}B$. Hence by (5.12)

$$(\overline{a}, \overline{b}, \overline{\xi})^2 = (\overline{a}, \overline{b})(\overline{a}, \overline{b}, \overline{\xi})$$

in B, where $\overline{*}$ denotes the reduction mod (x). Therefore

$$(a, b, \xi)^2 \subseteq (a, b)(a, b, \xi) + (x)$$

whence

$$(a, b, \xi)^2 = (a, b)(a, b, \xi) \tag{5.13}$$

because $(a, b, \xi) \subseteq \mathfrak{p}$ and $(x) \cap \mathfrak{p} = (0)$. Since $\xi^2 \in (a, b)(a, b, \xi) = (x^n + a, b)(a, b, \xi) \subseteq QI$ by (5.13) and $x^{2n} = (x^n + a)x^n \in QI$, we get that $(x^n, \xi)^2 \subseteq QI$, and so $I^2 = QI$ because $I^2 = QI + (x^n, \xi)^2$ (cf. (5.12)). Thus $I^2 = QI$, if $QB \neq (QB)^\sharp$. Conversely, assume that $I^2 = QI$. Then $IB \subseteq (QB)^\sharp$, whence $QB \neq (QB)^\sharp$ because $QB \subsetneq IB = QB : \mathfrak{m}B \subseteq (QB)^\sharp$. Thus $I^2 = QI$ if and only if $QB \neq (QB)^\sharp$, provided $\ell = 1$ and $\ell_A(I/Q) = 2$. This completes the proof of Theorem (5.3). $\square$

Corollary (5.14). *Let $\ell = 1$ and $\ell_A(I/Q) = 2$. Then $I \subseteq Q^\sharp$ if and only if $QB \neq (QB)^\sharp$. When this is the case, the equality $I^2 = QI$ holds true.*

Proof. Suppose that $QB = (QB)^\sharp$ and $I \subseteq Q^\sharp$. Then $IB = QB$, so that the monomorphism $A/(\mathfrak{p} + Q) \to A/Q$ in (5.4) has to be bijective on the socles, whence $\ell_A(I/Q) = 1$. This is impossible. If $QB \neq (QB)^\sharp$, we get by Theorem (5.3) that $I^2 = QI$ whence $I \subseteq Q^\sharp$. $\square$

Assume that $\ell = 1$ and let $Q = (x - y, y^2 - z^2)$. Then $\ell_A(I/Q) = 2$. We have by (5.14) $I \not\subseteq Q^\sharp$, since $QB = (QB)^\sharp$ (cf. Theorem (1.1)). This shows the equality $I^2 = QI$ does not necessarily hold true when $\ell = 1$.

Secondly, let $\mathfrak{a} = (X^3, XY, Y^2 - XZ)$ and let $A = R/\mathfrak{a}$. Let x, y and z denote the reduction of X, Y and Z mod $\mathfrak{a}$. Let $\mathfrak{p} = (x, y)$. We then have the following.

Lemma (5.15). *A is a Buchsbaum local ring with* $\dim A = 1$, $\mathrm{H}^0_\mathfrak{m}(A) = (x^2) \neq (0)$, *and* $\mathrm{e}(A) = \mathrm{r}(A) = 3$.

Proof. We have $\sqrt{\mathfrak{a}} = (X, Y)$, whence $\dim A = 1$ and $\mathrm{Min} A = \{\mathfrak{p}\}$. We certainly have that $\mathfrak{m}x^2 = (0)$ and $x^2 \neq 0$. Thus $(x^2) \subseteq \mathrm{H}^0_\mathfrak{m}(A)$. Let

$$B = A/(x^2) \cong R/(X^2, XY, Y^2 - XZ).$$

We will show that B is a Cohen-Macaulay ring with $\mathrm{e}(B) = 3$. Let $\mathfrak{b} = (X^2, XY, Y^2 - XZ)$ and $P = (X, Y)$. Then $P = \sqrt{\mathfrak{b}}$, $PR_P = (X - \frac{Y^2}{Z}, Y)R_P$, and $\mathfrak{b}R_P = (X - \frac{Y^2}{Z}, Y^3)R_P$. Hence $\mathrm{e}(B) = \ell_{R_P}(R_P/\mathfrak{b}R_P) = 3$, because R/P is a DVR. Since $\mathfrak{n}^2 = Z\mathfrak{n} + \mathfrak{b}$, the ideal zB is a minimal reduction of the maximal ideal $\mathfrak{n}/\mathfrak{b}$ in B, so that we have $\mathrm{e}^0_{zB}(B) = \mathrm{e}(B) = 3$, while $\ell_B(B/zB) = \ell_R(R/(X^2, XY, Y^2, Z)) = 3$. Thus $\ell_B(B/zB) =$

$\mathrm{e}^0_{zB}(B) = 3$, whence $B = A/(x^2)$ is a Cohen-Macaulay ring and $\mathrm{H}^0_{\mathfrak{m}}(A) = (x^2)$. Let $a \in \mathfrak{m}$ be a parameter in A. Then $(0) : a \subseteq \mathrm{H}^0_{\mathfrak{m}}(A) = (x^2)$, since a is a non-zerodivisor in the Cohen-Macaulay ring $B = A/\mathrm{H}^0_{\mathfrak{m}}(A)$. Hence $\mathfrak{m}{\cdot}[(0) : a] = (0)$, so that A is a Buchsbaum ring. We have $\mu_{\hat{A}}(\mathrm{K}_{\hat{A}}) = \mu_{\hat{B}}(\mathrm{K}_{\hat{B}}) = \mathrm{r}(B) = 2$, because $\mathrm{H}^1_{\mathfrak{m}}(A) \cong \mathrm{H}^1_{\mathfrak{m}}(B)$ and $(X^2, XY, Y^2, Z) : \mathfrak{n} = \mathfrak{n}$. Hence $\mathrm{r}(A) = \ell_A(\mathrm{H}^0_{\mathfrak{m}}(A)) + \mathrm{r}(B) = 1 + 2 = 3$. $\square$

Let $Q = (a)$ be a parameter ideal in A and put $I = Q : \mathfrak{m}$. Since $A/\mathfrak{p}$ is a DVR with z mod $\mathfrak{p}$ a regular parameter, we may write $a = \varepsilon z^n + b_0$ with $\varepsilon \in \mathrm{U}(A)$, $n \geq 1$, and $b_0 \in \mathfrak{p}$. Hence $Q = (z^n + b)$, where $b = \varepsilon^{-1} b_0 \in \mathfrak{p}$. Consequently, letting $b = xf + yg$ with $f, g \in A$, we may assume from the beginning that

$$a = z^n + xf + yg \quad \text{and} \quad Q = (a). \tag{5.16}$$

With this notation we have the following.

Theorem (5.17). *The equality $I^2 = QI$ holds true if and only if one of the following conditions is satisfied.*

(1) $f \notin \mathfrak{m}$.

(2) $f \in \mathfrak{m}$ *and* $n > 1$.

We have $I^3 = QI^2$ but $I^2 \neq QI$, if $f \in \mathfrak{m}$ and $n = 1$.

Proof. (1) If $f \notin \mathfrak{m}$, then A/Q is a Gorenstein ring and $I = Q + (x^2)$. In fact, choose $F, G \in R$ so that f, g are the reductions of F, G mod $\mathfrak{a}$, respectively. Then $F \notin \mathfrak{n}$. We put $V = Z^n + XF + YG$ and $\mathfrak{q} = (V, XY, Y^2 - XZ)$. Then $\sqrt{\mathfrak{q}} = \mathfrak{n}$ and so $\mathfrak{q}$ is a parameter ideal in R. Let x, y, and z be, for the moment, the reductions of X, Y, and Z mod $\mathfrak{q}$. We put $\xi = -F$ mod $\mathfrak{q}$ and $\eta = G$ mod $\mathfrak{q}$. Then since $x\xi = z^n + y\eta$, we have

$$\begin{aligned}
(x\xi)^3 &= (z^n + y\eta)(x\xi)^2 \\
&= z^n (x\xi)^2 \qquad (\text{since} \ \ xy = 0) \\
&= (x\xi{\cdot}z)(x\xi)z^{n-1} \\
&= (y^2\xi)(x\xi)z^{n-1} \qquad (\text{since} \ \ y^2 = xz) \\
&= 0.
\end{aligned}$$

Thus $x^3 = 0$ in $R/\mathfrak{q}$. Consequently $X^3 \in \mathfrak{q}$, so that $\mathfrak{q} = (V, X^3, XY, Y^2 - XZ)$. Hence $A/Q = A/(z^n + xf + yg) \cong R/(V, X^3, XY, Y^2 - XZ) = R/\mathfrak{q}$ and so A/Q is a Gorenstein ring. Since $\ell_A(I/Q) = 1$ and $x^2 \notin Q$ (otherwise, $x^2 \in \mathrm{H}^0_{\mathfrak{m}}(A) \cap Q = (0)$; recall that A is a Buchsbaum ring), we get that $I = Q + (x^2)$. Thus $I^2 = QI$.

(2) Suppose that $f \notin \mathfrak{m}$ and $n > 1$. Then, since $xa = xz^n$ and $ya = yz^n + y^2 g = yz^n + xzg$, we get

$$a\mathfrak{p} = (xz^n, yz^n + y^2 g) \subseteq (z) \tag{5.18}$$

and $\mathfrak{m}\cdot(xz^{n-1}, x^2) \subseteq a\mathfrak{p}$. We claim that the reductions of xz^{n-1} and x^2 mod $a\mathfrak{p}$ are linearly independent in $\mathfrak{p}/a\mathfrak{p}$ over the field $A/\mathfrak{m}$. In fact, let $c_1, c_2 \in A$ and assume that $c_1(xz^{n-1}) + c_2x^2 \in a\mathfrak{p}$. Then since $n > 1$ and $a\mathfrak{p} \subseteq (z)$ by (5.18), we have $c_2x^2 \in (z)$, and so $c_2x^2 \in \mathrm{H}^0_{\mathfrak{m}}(A) \cap (z) = (0)$ (recall that (z) is a parameter ideal in A). Hence $c_2 \in \mathfrak{m}$ so that $c_1(xz^{n-1}) \in a\mathfrak{p}$. Suppose $c_1 \notin \mathfrak{m}$ and write $xz^{n-1} = xz^n\varphi + (yz^n + y^2g)\psi$ with $\varphi, \psi \in A$. Then because $xz^{n-1}(1 - z\varphi) = (yz^n + y^2g)\psi$, we get $xz^{n-1} = (yz^n + y^2g)\rho$ for some $\rho \in A$. Hence

$$z^{n-1}(x - yz\rho) = y^2g\rho = xzg\rho. \tag{5.19}$$

Now notice that $A/(x) \cong R/(X, Y^2)$ and we see that z is $A/(x)$-regular. Because $z^{n-1}(-yz\rho) \equiv 0 \mod (x)$ (cf. (5.19)), we get $y\rho \equiv 0 \mod (x)$, whence $y^2\rho = 0$. This implies by (5.19) that

$$x - yz\rho \in (0) : z^{n-1} = (0) : z = (x^2)$$

since z is a parameter in our Buchsbaum ring A. Thus $x \in \mathfrak{m}^2$ which is impossible. Hence $c_1 \in \mathfrak{m}$.

Now let $B = A/\mathfrak{p}$ and look at the canonical exact sequence

$$0 \to \mathfrak{p}/a\mathfrak{p} \to A/Q \to B/QB \to 0 \tag{5.20}$$

of A-modules and we have

$$2 \leq \ell_A((0) :_{\mathfrak{p}/a\mathfrak{p}} \mathfrak{m}) \leq \ell_A(I/Q) \leq \mathrm{r}(A) = 3. \tag{5.21}$$

If $\ell_A(I/Q) = \mathrm{r}(A) = 3$, then $I^2 = QI$ by Theorem (3.9). Hence to prove $I^2 = QI$, we may assume $\ell_A(I/Q) \leq 2$. Therefore $\ell_A((0) :_{\mathfrak{p}/a\mathfrak{p}} \mathfrak{m}) = \ell_A(I/Q) = 2$ by (5.21) so that by (5.20) we have $I = Q + (xz^{n-1}, x^2)$, because $[(0) :_{\mathfrak{p}/a\mathfrak{p}} \mathfrak{m}]$ is generated by the reductions of xz^{n-1} and x^2 mod $a\mathfrak{p}$. Hence $I^2 = QI + (xz^{n-1}, x^2)^2 = QI$, since $x^2\mathfrak{m} = (0)$.

(3) Suppose that $f \in \mathfrak{m}$ and $n = 1$. Let $f = xf_1 + yf_2 + zf_3$ with $f_i \in A$. Then $a = z + xf + yg = z + x^2f_1 + y(g + yf_3)$, because $y^2 = xz$. Consequently, replacing f by xf_1 and g by $g + yf_3$, we may assume in the expression (5.16) of I that

$$a = z + x^2f + yg \quad \text{and} \quad Q = (a).$$

Hence $a\mathfrak{p} = (xz, yz + y^2g) = (xz, yz) = z\mathfrak{p}$ (recall that $y^2 = xz$). Look at the exact sequence

$$0 \to \mathfrak{p}/a\mathfrak{p} \to A/(z) \to B/zB \to 0 \tag{5.22}$$

of A-modules. Then, because $A/(z) \cong R/(X^3, XY, Y^2, Z)$, we see $\ell_A(((z) : \mathfrak{m})/(z)) = 2$ and $(z) : \mathfrak{m} = (z) + (x^2, y) \subseteq (z) + \mathfrak{p}$. Hence in (5.22) the canonical epimorphism $A/(z) \to B/zB$ is zero on the socles. Thus $\ell_A((0) :_{\mathfrak{p}/a\mathfrak{p}} \mathfrak{m}) = 2$ and $[(0) :_{\mathfrak{p}/a\mathfrak{p}} \mathfrak{m}]$ is generated by the reductions of x^2 and y mod $a\mathfrak{p} = z\mathfrak{p}$. Consequently $Q + (x^2, y) \subseteq I$ by (5.20).

Claim (5.23). $\ell_A(I/Q) \neq 3$.

Proof of Claim (5.23). Assume $\ell_A(I/Q) = 3$. Then $I^2 = QI$ by Theorem (3.9), since $\ell_A(I/Q) = \mathrm{r}(A)$. Thus $IB = QB$, because $IB \subseteq (QB)^\sharp = QB$ (notice that B is a DVR). Hence in (5.20) the epimorphism $A/Q \to B/QB$ has to be zero on the socles, and so $\ell_A(I/Q) = \ell_A((0) :_{\mathfrak{p}/a\mathfrak{p}} \mathfrak{m}) = 2$, which is impossible. $\square$

By this claim we see that $I = Q + (x^2, y)$, whence $I^2 = QI + (y^2)$. Consequently, $I^3 = QI^2$, because $y^3 = y{\cdot}xz = 0$. In contrast, $I^2 \neq QI$, because $y^2 \notin QI$. To see this, assume that $y^2 \in QI$ and choose $F, G \in R$ so that f, g are the reductions of F, G mod $\mathfrak{a}$, respectively. Let $K = (Z^2 + YZG, YZ + Y^2G, X^3, XY, Y^2 - XZ)$. Then $Y^2 \in K$, because $QI = (z + x^2f + yg)(z, x^2, y) = (z^2 + yzg, yz + y^2g)$. Hence

$$K = (X^3, Y^2, Z^2, XY, YZ, ZX)$$

which is impossible, since $\mu_R((X^3, Y^2, Z^2, XY, YZ, ZX)) = 6$ while $\mu_R(K) \leq 5$. Thus $y^2 \notin QI$, which completes the proof of Theorem (5.17). $\square$

If $Q \subseteq \mathfrak{m}^2$, then $n \geq 2$, and so by Theorem (5.17) we readily get the following.

Corollary (5.24). *$I^2 = QI$ if $Q \subseteq \mathfrak{m}^2$.*

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